State Aggregation with α-MEU*

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This paper studies how ambiguity affects identification of the state aggregation model – a model with an agent that combines several states into an event in order to simplify decision-making. We show that the state aggregation is partially identified under Maxmin Expected Utility (MEU) and it is uniquely identified if α-MEU model is used. In addition, we offer testable restrictions of the model.

1 Introduction

Burkovskaya (2017) introduced the model of state aggregation where an agent combines several states into an event in order to simplify decision-making. She proposed State Aggregation Subjective Expected Utility (SASEU) – a model where agents have subjective probabilities of states and evaluate acts as an expected utility on aggregated events. The author shows that SASEU allows identification of the state aggregation from observable choices of Arrow securities. However, a vast literature (Ellsberg (1961), Kahneman and Tversky (1979), etc.) demonstrates that in the presence of ambiguity - i.e., when agents do not have unique subjective probabilities - people make choices that are not consistent with SEU. Hence, the main question of this paper is whether the state aggregation can be still identified if the agent is allowed to have a set of multiple priors instead of single probabilities of states.

Home insurance is a good example where state aggregation might happen. There are three states of the world: robbery, flood and no adversity. A person is deciding on home insurance in

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this case and the policy booklet suggests: "Unexpected comes in different forms. Protect yourself against all of them!" The agent is nudged to think about robbery and flood together. Hence, she combines flood and robbery into event "adversity." Now instead of three separate states of the world, the state space is partitioned into "adversity" and "no adversity." As a result, the individual smoothes consumption between events and between states inside each event instead of between all of the states. The last affects the choice of the insurance plan.

In this paper, we (1) show that under full-dimensionality of priors and strict α-MEU ambiguity model, preferences imply a unique state aggregation; (2) demonstrate that under MEU model behavior with any state aggregation can be also produced by the model without the state aggregation, but it would require rectangular priors; (3) identify the state aggregation from choices; and, (4) suggest testable restrictions of the model.

First, we find that if all sets of priors are full-dimensional, then preferences with a strict (i.e., $0 < \alpha < 1$) α-MEU stage model imply a unique state aggregation. This means that different state aggregations generate distinct preference relations. We also show that in the case of the MEU stage model, full identification is possible only when the agent does not aggregate states and has non-rectangular priors. When priors satisfy rectangularity, a property that provides dynamic consistency of MEU (Epstein and Schneider (2003)), we are able to achieve only partial identification: it is not possible to distinguish between the whole state space with rectangular priors and the aggregation into a partition, for which rectangularity of priors holds. Note that it opens a new perspective for dynamic consistency – a property that is usually desired: lack of dynamic consistency allows to identify the state aggregation, the frame in the agent's mind.

Second, we show identification of the state aggregation from choices and prices in a complete market. We make an additional assumption of polytope priors that allows for SEU behavior in different regions of the act space. This type of behavior implies that the agent uses the same subjective probabilities in order to evaluate an act in some region; however, these probabilities differ across regions. The described property permits construction of a method of identification that might be adapted to estimate the model in the future. The proposed method consists of two steps: (1) to identify region-specific probabilities together with utility function; and (2) to obtain the state aggregation from the way that these probabilities change across the regions.

Finally, we provide an Afriat-like finite system of inequalities that allows for testing our model by a dataset. Since the system is bilinear, the Tarski-Seidenberg algorithm can be applied to determine whether the solution exists. An example of the potential dataset is data on choices of home insurance plans that consist of deductibles and income.

Section 2 provides the basic notation of α-MEU preferences and a model of state aggregation. Section 3 consists of uniqueness results. Section 4 discusses the identification of the state aggregation from choices and prices in the complete market setting. Testable implications of the model are provided in Section 5. Appendix A includes related axioms and the representation theorem. All other proofs are in Appendix B.

Related Literature

First of all, the paper is related to the literature on the state aggregation. Burkovskaya (2017) introduced the model and axioms, and demonstrated that when the stage model is a non-affine transformation of SEU the state aggregation is identified. We should that t is still identified if the stage model is a strict α-MEU, however, it is only partially identified with MEU.

Second, our work is related to the literature on dynamic consistency. The relationship between non-expected utility and problems with dynamic consistency is well known (Machina (1989), Karni and Schmeidler (1991), Epstein and LeBreton (1993), Ghirardato (2002), etc.). Our results are closely related to those of Epstein and Schneider (2003), who find that the rectangularity of priors is an important condition for dynamic consistency of the MEU model. However, we find that a lack of dynamic consistency often allows for identification of the unique state aggregation. That is why we can uniquely identify it under the MEU stage model only when the set of priors is not rectangular. In addition, a strict α-MEU model with full dimensional priors is generally dynamically inconsistent and, as a result, provides full identification. We expect similar results to hold for other ambiguity stage models.

Finally, this paper is also related to the revealed preference field. The literature, started by Afriat (1967), suggests that a dataset can be rationalized by an increasing concave continuous utility function if and only if there exists a solution to a system of linear inequalities obtained from the dataset. Similar tests for expected utility and its generalizations were developed afterwards (Varian (1983), Diewert (2012), Kubler, Selden, and Wei (2014), Echenique and Saito (2015), etc.). Polisson, Quah, and Renou (2015) develop a testing procedure that can be used for many different models of choice under uncertainty. They allow for non-concave utility, and, as a result, test for an actual model without extra assumptions. However, the price of no additional assumptions is that the system of inequalities becomes non-linear. The procedure from Polisson et al. is applicable to the model developed in this paper after additional inequalities on priors are derived. The final system of inequalities for testing the model is finite and bilinear. The previous literature (e.g. Brown and Matzkin (1996), etc.) suggests applying the Tarski-Seidenberg algorithm in this case.

2 Notation and the model

Suppose that X is a convex subset of consequences in $\mathbb{R}$, and Ω is a finite set of states of the world with an algebra Σ of subsets of Ω. We denote $\mathcal{F}$ a set of all acts, Σ-measurable finite step functions: $\Omega \to \Delta X$. Let $\mathcal{M}$ be a set of elements π, such that $\pi \subset \Sigma$ is a partition of Ω. Partitions of the state space represent different ways of state aggregation.

2.1 α-MEU preferences

As stage preferences, we assume the α-MEU model of Ghirardato, Maccheroni and Marinacci (2004). If $\alpha \in [0,1]$ is a parameter of ambiguity aversion, $u(\cdot)$ is an increasing continuous utility function, $\mathcal{P}$ is a set of priors, and $x \in \mathcal{F}$ is an act, then the value of x is assessed by

$$V(x) = \alpha \min_{P \in \mathcal{P}} \left[\sum_{s \in \Omega} u(x(s)) P(s) \right] + (1-\alpha) \max_{P \in \mathcal{P}} \left[\sum_{s \in \Omega} u(x(s)) P(s) \right]$$

If we denote $P_{\min x}(s)$ and $P_{\max x}(s)$ to be probabilities that minimize and maximize "expected" utility, then

$$V(x) = \sum_{s \in \Omega} u(x(s)) \left(\alpha P_{\min x}(s) + (1-\alpha) P_{\max x}(s) \right) = \sum_{s \in \Omega} u(x(s)) P_{\alpha;x}(s).$$

2.2 State Aggregation

A stage in which several states are combined into one event will be called the conditional stage, and the set of priors $\mathcal{P}(A)$ used for aggregation into event A is the set of conditional priors. We will also call the ex-ante stage (or problem) a stage in which evaluation of an act across events happens. Given the subjective partition π, the set of priors over events $\mathcal{P}(\pi)$ will be called the set of ex-ante priors.

Definition 1. *The agent's behavior is said to exhibit α-MEU State Aggregation Representation (α-MEU SAR) if there exist a partition of the state space π, nonempty weak compact and convex sets $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$ for any event $A \in \pi$ of probabilities on Σ, and a continuous monotone function $u : X \to \mathbb{R}$, $\alpha \in [0,1]$ such that the agent optimizes the functional $V(\cdot|\pi)$:*

$$V(x|A) = \alpha \min_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(x(s))P(s|A) + (1-\alpha) \max_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(x(s))P(s|A)$$

$$V(x|\pi) = \alpha \min_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} V(x|A)P(A|\pi) + (1-\alpha) \max_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} V(x|A)P(A|\pi).$$

A set of axioms for the above representation is provided in Section 6. Definition 1 implies that the agent evaluates each act given a partition by a folding-back procedure: First, she aggregates states into event A and evaluates the act x given every such event A in the partition π with the set of conditional priors $\mathcal{P}(A)$:

$$V(x|A) = \alpha \min_{P(\cdot|A)\in\mathcal{P}(A)} \left[\sum_{s\in A} u(x(s))P(s|A)\right] + (1-\alpha) \max_{P(\cdot|A)\in\mathcal{P}(A)} \left[\sum_{s\in A} u(x(s))P(s|A)\right],$$

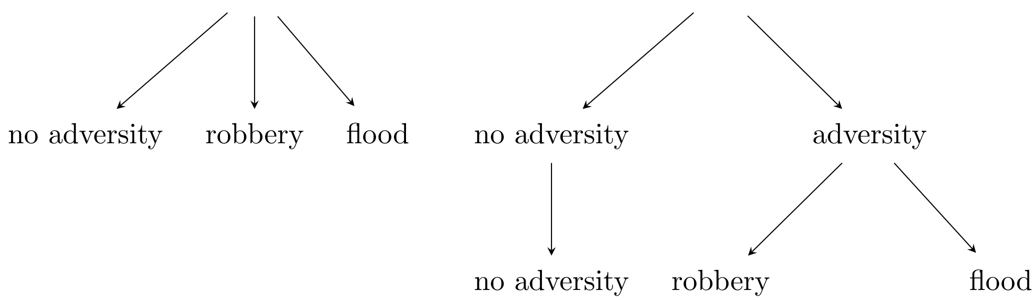
where α is an agent's coefficient of ambiguity aversion. Second, the agent evaluates the act across events that form her subjective partition. Thus, at the ex-ante stage with the set of ex-ante priors $\mathcal{P}(\pi)$, the agent's value of the act is obtained similarly to the way it is obtained in the conditional stage:

$$V(x|\pi) = \alpha \min_{P(\cdot|\pi)\in\mathcal{P}(\pi)} \left[\sum_{A\in\pi} V(x|A)P(A|\pi)\right] + (1-\alpha) \max_{P(\cdot|\pi)\in\mathcal{P}(\pi)} \left[\sum_{A\in\pi} V(x|A)P(A|\pi)\right].$$

Example 1. The agent chooses a home insurance policy under three states of the world:

natural disaster, robbery, and no accident.

Suppose that the probability of no adversity (s_1) is not smaller than 50%. However, the probabilities of robbery (s_2) and flood (s_3) are still unknown. Imagine two different situations: (1) the agent does not aggregate states, and her state aggregation $\pi_0 = \{s_1, s_2, s_3\}$ is the whole outcome space; and (2) the agent has difficulty optimizing over three states, and she splits the whole outcome space into events "no adversity" $A_1 = \{s_1\}$ and "adversity" $A_2 = \{s_2, s_3\}$. We denote the state aggregation in this case as $\pi = \{A_1, A_2\}$.



In situation (1), the ex-ante stage implies regular evaluation over the whole state space. Then, the set of ex-ante priors is $\mathcal{P}(\pi_0) = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 \geq 0.5; \sum p_i = 1\}$. All conditional stages are degenerate in this case because each event consists of exactly one state.

In situation (2), the agent's state aggregation π consists of two events, A_1 and A_2. In addition, consider that the agent updates prior-by-prior using Bayes' rule. Thus, the set of ex-ante priors is $\mathcal{P}(\pi) = \{(p_{A_1}, p_{A_2}) \in \mathbb{R}^2 : p_{A_1} \geq 0.5; p_{A_1} + p_{A_2} = 1\}$. In the case of no adversity, the set of conditional on event A_1 priors is degenerate: $\mathcal{P}(A_1) = \{p_1 \in \mathbb{R} : p_1 = 1\}$. If the event is "adversity," then the set of conditional on event A_2 priors is $\mathcal{P}(A_2) = \{(p_2, p_3) \in \mathbb{R}^2 : p_2 + p_3 = 1\}$.

Now, for simplicity, consider utility function $u(x) = x$ and a parameter of ambiguity aversion $0 < \alpha < 1$. First, imagine the agent without state aggregation. Then, her value of act $x = (x_1, x_2, x_3)$ that represents the amount of money/consumption in every state is

$$V(x|\pi_0) = \frac{1}{2}x_1 + \frac{\alpha}{2}\min_i(x_i) + \frac{1-\alpha}{2}\max_i(x_i).$$

In case (2), the value of act x at event A_2 is

$$V(x|A_2) = \alpha \min(x_2, x_3) + (1-\alpha)\max(x_2, x_3),$$

while the value of act x under state aggregation π is

$$V(x|\pi) = \frac{1}{2}x_1 + \frac{\alpha}{2}\min(x_1, V(x|A_2)) + \frac{1-\alpha}{2}\max(x_1, V(x|A_2)).$$

As long as $0 < \alpha < 1$, $V(x|\pi_0) \neq V(x|\pi)$. To easily see this, consider, for example, x such that $x_1 > x_2 > x_3$. Then, $V(x|\pi_0) = \frac{2-\alpha}{2}x_1 + \frac{\alpha}{2}x_3$, while $V(x|\pi) = \frac{2-\alpha}{2}x_1 + \frac{\alpha(1-\alpha)}{2}x_2 + \frac{\alpha^2}{2}x_3$. Thus, with the state aggregation π, preferences differ from the optimal. Notice also that if $\alpha = 0, 1$, then $V(x|\pi_0) = V(x|\pi)$.

3 Uniqueness

This section discusses conditions when preferences $\succeq$ imply a unique state aggregation. It is important to note that the whole state space partition Ω and a trivial partition $\{\Omega\}$, which consists of one event that includes all states, always result in the same preferences, and, as a result, it is not possible to distinguish between them. From now on, we will treat them as one partition π_0.

As the above example demonstrates, strict α-MEU and MEU models might provide different results when it comes to identification. The two following subsections discuss uniqueness under these models separately.

3.1 Strict α-MEU

Assumption 1. *For any event $A \in \pi$, the sets of priors $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$ are full-dimensional.*

The differences in preferences that arise due to distinct state aggregations come from various ways of bundling ambiguity. If assumption 1 does not hold, we know that some information is not ambiguous. Thus, we might get different state aggregations depending on which states we bundle together. In order to see this, consider the following example:

Example 2. Suppose that the objective probability of no accident is 0.15. The agent's state aggregation is $\pi = \{A_1, A_2\}$, where $A_1 = \{s_1\}$ (no accident) and $A_2 = \{s_3, s_2\}$ (accident). The set of priors $\mathcal{P}(\Omega)$ is such that $P(s_3|\Omega) + P(s_2|\Omega) = 0.85$ and $P(s_1|\Omega) = 0.15$. Notice that $\mathcal{P}(\Omega)$ is not a full dimensional polytope. Let's compare behavior under state aggregation π with no state aggregation π_0. The probability of no accident is objective and does not depend on act x itself $P_x(s_1|\pi) = P_x(s_1|\pi_0) = 0.15$. We denote payoff at state i as $x_i = x(s_i)$; then, in order to evaluate an act, the agent without state aggregation has to solve the following problem:

$$\pi_0: \ P(s_1)u(x_1) + P(s_2)u(x_2) + P(s_3)u(x_3) \to \max / \min$$

$$\text{s.t. } P(s_1) = 0.15; \ P(s_2) + P(s_3) = 0.85.$$

The agent with subjective partition π will have to split the problem into two tasks. First, the conditional stage:

$$P(s_2|A_2)u(x_2) + P(s_3|A_2)u(x_3) \to \max / \min$$

$$\text{s.t. } P(s_2|A_2) + P(s_3|A_2) = 1.$$

Second, the ex-ante stage, which is degenerate in this case:

$$P(s_1)u(x_1) + P(A_2)V(x|A_2) \to \max / \min$$

$$\text{s.t. } P(s_1) = 0.15; \ P(A_2) = 0.85.$$

Note that the problem of the agent without state aggregation is equivalent to the set of problems with the state aggregation π. Thus, $P_x(s_i|\pi) = P_x(s_i|\pi_0)$. This implies that $V(x|\pi) = V(x|\pi_0)$ for any act $x \in \mathcal{F}$. It happens because unambiguous state s_1 might be put together or separated from all other states as long as it does not affect the bundling of ambiguity.

Proposition 1. *Suppose that* $0 < \alpha < 1$*, assumption 1 holds, and* $\pi, \pi' \in \mathcal{A}$ *both deliver* α*-MEU SAR of preferences* $\succeq$*; then,* $\pi = \pi'$*.*

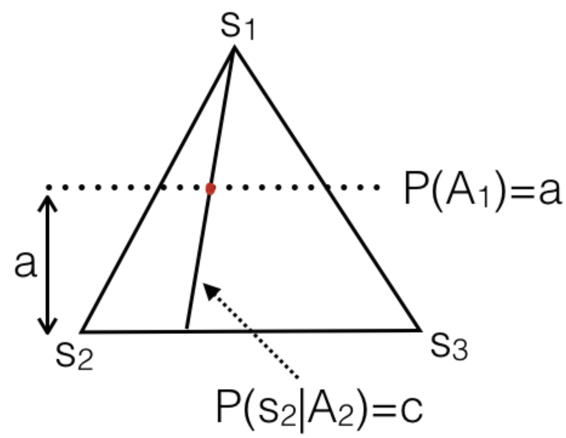


Figure 1: Probability simplex

3.2 MEU

Under the MEU model, full identification is not always possible. Rectangularity of priors is an important property for identification in this case.

Definition 2. *The set of priors $\mathcal{P}(\pi_0)$ is called π-rectangular if and only if there is a set $\mathcal{P}(\pi)$ and for any event $A \in \pi$, there exist sets $\mathcal{P}(A)$ such that for any state $s \in A$:*

$$\mathcal{P}(\pi_0)(s) = \{p_s p_A : p_s \in \mathcal{P}(A)(s), p_A \in \mathcal{P}(\pi)(A)\} = \mathcal{P}(A)(s)\mathcal{P}(\pi)(A).$$

It is easy to understand rectangularity graphically in a probability simplex. Let $\Omega = \{s_1, s_2, s_3\}$ and partition $\pi = \{A_1, A_2\}$, where $A_1 = s_1$ and $A_2 = \{s_2, s_3\}$. A point in the triangle represents one prior: the probability of state is the distance from the point to the side opposite to the vertex corresponding to the state. Now, suppose that we are interested in showing, conditional on event A_2, probabilities in the simplex. First, if $P(A_1) = a$, then all priors that satisfy it can be represented in the simplex as a line parallel to the side $s_2 s_3$ at distance a from it. In this case, $P(s_2|A_2) = c$ is the point such that the ratio of the right part of the line $P(A_1) = a$ to the whole line is c (see Figure 1). Thus, the conditional probability splits the line of $P(A_1) = a$ in a given ratio. Next, we allow the probability of event A_1 to vary from 0 to 1. In this case, $P(s_2|A_2) = c$ is a line that starts in s_1 and ends at the side $s_2 s_3$.

Now we consider what rectangular priors look like in the simplex. The definition specifies

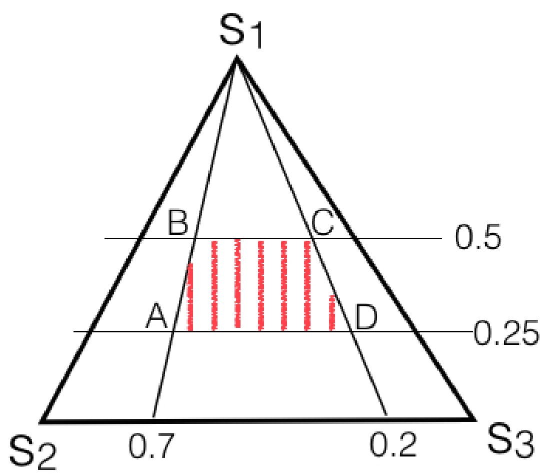


Figure 2: Rectangular priors

that a rectangular set can be obtained from the multiplication of conditional and ex-ante sets of priors. For example, suppose that $\mathcal{P}(\pi) = \{(p_{A_1}, p_{A_2}) \in \mathbb{R}^2 : 0.25 \le p_{A_1} \le 0.5;\ p_{A_1} + p_{A_2} = 1\}$ and $\mathcal{P}(A_2) = \{(p_2, p_3) \in \mathbb{R}^2 : 0.2 \le p_2 \le 0.7;\ p_2 + p_3 = 1\}$. Then, ABCD in Figure 2 is the π-rectangular set of priors that is obtained from the intersection of the sets $\mathcal{P}(\pi)$ and $\mathcal{P}(A_2)$.

Rectangularity is the reason for dynamic consistency for MEU in Epstein and Schneider (2003). In contrast, we find that a lack of dynamic consistency allows for the identification of the state aggregation. Unfortunately, with MEU, behavior under any state aggregation π could be also modeled by the whole state space as the state aggregation with π-rectangular priors. Thus, in this situation, we can obtain only partial identification.

Proposition 2. *Suppose that $\alpha = 1$, and a state aggregation π provides α-MEU SAR of preferences $\succeq$; then, the preferences can also be represented by partition π_0 with π-rectangular priors.*

However, when the state aggregation is the whole state space and priors are non-rectangular, we can uniquely identify the partition.

Proposition 3. *Suppose that $\alpha = 1$, assumption 1 holds, π_0, π both provide α-MEU SAR of preferences $\succeq$, and priors $\mathcal{P}(\pi_0)$ are not π-rectangular; then, $\pi = \pi_0$.*

4 Identification in the market

In this section, we show how to identify the state aggregation from choices of Arrow securities and their prices. Even if Arrow assets are not directly available in the market, as long as the market is complete, Arrow prices can always be uniquely recovered.

4.1 Notation

Let us define some notation that is used below. Given event A and act $x \in \mathcal{F}$, we denote the conditional priors that minimize and maximize "expected" utility $\sum_{s\in A} u(x(s))P(s|A)$ at the conditional stage, where states are aggregated into event A, as follows:

$$P_{\min x}(\cdot|A) = \underset{P(\cdot|A)\in\mathcal{P}(A)}{\operatorname{argmin}} \sum_{s\in A} u(x(s))P(s|A)$$

$$P_{\max x}(\cdot|A) = \underset{P(\cdot|A)\in\mathcal{P}(A)}{\operatorname{argmax}} \sum_{s\in A} u(x(s))P(s|A).$$

Similarly, given state aggregation π, the ex-ante priors that minimize and maximize "expected" value at the ex-ante stage are:

$$P_{\min x}(\cdot|\pi) = \underset{P(\cdot|\pi)\in\mathcal{P}(\pi)}{\operatorname{argmin}} \sum_{A\in\pi} V(x|A)P(A|\pi)$$

$$P_{\max x}(\cdot|\pi) = \underset{P(\cdot|\pi)\in\mathcal{P}(\pi)}{\operatorname{argmax}} \sum_{A\in\pi} V(x|A)P(A|\pi).$$

To simplify the notation, we denote

$$P_{\alpha;x}(A|\pi) = \alpha P_{\min x}(A|\pi) + (1-\alpha)P_{\max x}(A|\pi)$$

$$P_{\alpha;x}(s|A) = \alpha P_{\min x}(s|A) + (1-\alpha)P_{\max x}(s|A)$$

$$P_x(s|\pi) \equiv P_{\alpha;x}(s|A)P_{\alpha;x}(A|\pi).$$

where $s \in A \in \pi$. We abuse the notation by calling $P_{\alpha;x}(A|\pi)$ and $P_{\alpha;x}(s|A)$ the probabilities of ex-ante and conditional stages, where no confusion should arise. We also call $P_x(s|\pi)$ a probability.[1]

4.2 Non-parametric example

Suppose that $\Omega = \{s_1, s_2, s_3\}$, partition $\pi = \{A_1, A_2\}$, where $A = \{s_1\}$ and $A_2 = \{s_2, s_3\}$, and sets of priors are $\mathcal{P}(A_2) = \{(p_2, p_3) \in \mathbb{R}^2 : p_3 \geq \phi,\ p_2 + p_3 = 1\}$ and $\mathcal{P}(\pi) = \{(p_{A_1}, p_{A_2}) \in \mathbb{R}^2 : p_{A_2} \geq \phi,\ p_{A_1} + p_{A_2} = 1\}$. Consider that a complete set of Arrow securities is available on the market.[2] The agent purchases a bundle of Arrow securities that maximizes her value given a certain amount of income I and the price p_i of an Arrow security that pays 1 in state i:

$$V((x_1, x_2, x_3)|\pi) \to \max_x$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 + p_3 x_3 = I.$$

Choices $x = (x_1, x_2, x_3)$, prices $p = (p_1, p_2, p_3)$ and income I are observed. We assume that all possible combinations of (p, I) are available. The purpose is to identify the state aggregation π, utility function $u(\cdot)$, sets of priors $\mathcal{P}(\pi)$ and $\mathcal{P}(A_2)$, and parameter α.

Note that the proposed set of priors is a full-dimensional polytope, so it implies that the "probability" of each state at a given act is a linear combination of solutions to some linear programming problems. Moreover, these probabilities are constant in some region around the act due to the fact that linear programming provides corner solutions. Thus, we can obtain these regions together with the related probabilities that the agent uses to solve the problem.

Consider the conditional on the event A_2 stage. This stage and its priors will produce one boundary $x_2 = x_3$, such that the regional conditional probabilities will be as follows:

1. If $x_2 > x_3$, then $P_{\min x}(s_3|A_2) = \phi$; $P_{\max x}(s_3|A_2) = 1$; $P_{x;\alpha}(s_3|A_2) = 1 - \alpha(1 - \phi)$.

2. If $x_2 < x_3$, then $P_{\min x}(s_3|A_2) = 1$; $P_{\max x}(s_3|A_2) = \phi$; $P_{x;\alpha}(s_3|A_2) = \alpha + (1 - \alpha)\phi$.

Now, consider the ex-ante stage. This stage boundary is $x_1 = u^{-1}\left(P_x(s_2|A_2)u(x_2) + P_x(s_3|A_2)u(x_3)\right)$. We denote the boundary as $b(x_2, x_3)$. This stage has the following ex-ante probabilities:

[1] Note that $P_x(s|\pi)$ behaves like a probability only for act x, but it is not a probability in a regular sense and changes across acts.

[2] An Arrow security pays one unit in a specified state and zero otherwise.

1. If $x_1 > b(x_2, x_3)$, then $P_{\min x}(A_2|\pi) = 1$; $P_{\max x}(A_2|\pi) = \phi$; $P_{f;\alpha}(A_2|\pi) = \alpha + (1-\alpha)\phi$.

2. If $x_1 < b(x_2, x_3)$, then $P_{\min x}(A_2|\pi) = \phi$; $P_{\max x}(A_2|\pi) = 1$; $P_{f;\alpha}(A_2|\pi) = \alpha\phi + 1 - \alpha$.

Thus, in total, we have four different regions with the following probabilities:

1. If $x_3 > x_2$ and $x_1 > b(x_2, x_3)$, then $P_x(s_1|\pi) = (1-\alpha)(1-\phi)$; $P_x(s_2|\pi) = (\alpha + (1-\alpha)\gamma)\alpha(1-\phi)$; $P_x(s_3|\pi) = (\alpha + (1-\alpha)\phi)(1-\alpha(1-\phi))$.

2. If $x_3 < x_2$ and $x_1 > b(x_2, x_3)$, then $P_x(s_1|\pi) = (1-\alpha)(1-\phi)$; $P_x(s_2|\pi) = (\alpha+(1-\alpha)\phi)(1-\phi)(1-\alpha)$; $P_x(s_3|\pi) = (\alpha + (1-\alpha)\phi)(\phi + (1-\phi)\alpha)$.

3. If $x_3 > x_2$ and $x_1 < b(x_2, x_3)$, then $P_x(s_1|\pi) = \alpha(1-\phi)$; $P_x(s_2|\pi) = (1-\alpha(1-\phi))\alpha(1-\phi)$; $P_x(s_3|\pi) = (1-\alpha(1-\phi))(1-\alpha(1-\phi))$.

4. If $x_3 < x_2$ and $x_1 < b(x_2, x_3)$, then $P_x(s_1|\pi) = \alpha(1-\phi)$; $P_x(s_2|\pi) = (1-\alpha(1-\phi))(1-\phi)(1-\alpha)$; $P_x(s_3|\pi) = (1-\alpha(1-\phi))(\phi + (1-\phi)\alpha)$.

The agents solves her maximization problem and chooses some bundle x:

$$P_x(s_1|\pi)u(x_1) + P_x(s_2|\pi)u(x_2) + P_x(s_3|\pi)u(x_3) \to \max_x$$
$$\text{s.t. } p_1x_1 + p_2x_2 + p_3x_3 = I.$$

For simplicity, assume that $\alpha > 0.5$ – i.e., the agent is ambiguity-averse. In this case, the indifference curves will have kinks at the bundles that connect different regions. Moreover, they will be observed as choices made under the same income, but under a range of prices. Thus, we will be able to separate the regions from each other in our observations.

Inside each region and for each chosen bundle and any states i and j, the optimality condition holds:

$$\frac{p_i}{p_j} = \frac{P_x(s_i|\pi)}{P_x(s_j|\pi)}\frac{u'(x_i)}{u'(x_j)}. \tag{1}$$

Now we analyze identification, for example, in region 1. For some fixed value of $\bar{x}_2$, we observe bundles with $x_3 > \bar{x}_2$ (and $x_1 > b_1(\bar{x}_2, x_3)$). By taking two bundles with different x_3 but the same $\bar{x}_2$ from this region:

$$\frac{p_3^1}{p_2^1} = \frac{P_x(s_3|\pi)}{P_x(s_2|\pi)}\frac{u'(x_3^1)}{u'(\bar{x}_2)} \text{ and } \frac{p_3^2}{p_2^2} = \frac{P_x(s_3|\pi)}{P_x(s_2|\pi)}\frac{u'(x_3^2)}{u'(\bar{x}_2)}$$

$$\frac{u'(x_3^2)}{u'(x_3^1)} = \frac{p_3^2 p_2^1}{p_2^2 p_3^1}.$$

Thus, the ratio of derivatives of the utility function can be identified in the interval $[\ \bar{x}_2; +\infty\ )$. Now, we choose another bundle with $\tilde{x}_2$, such that $\tilde{x}_2 > \bar{x}_2$, and $x_3 > \tilde{x}_2$, and some x_1 that satisfies conditions of region 1. The ratio of derivatives $\frac{u'(x_3)}{u'(\tilde{x}_2)}$ has already been identified, so the probability ratio is identified, as well:

$$\frac{P_x(s_3|\pi)}{P_x(s_2|\pi)} = \frac{p_3}{p_2}\frac{u'(\tilde{x}_2)}{u'(x_3)}.$$

By analogy, we can identify all probability ratios $\frac{P_x(s_i|\pi)}{P_x(s_j|\pi)}$ in region 1. Then, the probabilities can be recovered:

$$\frac{P_x(s_1|\pi)}{P_x(s_2|\pi)} = c_1; \text{ and } \frac{P_x(s_3|\pi)}{P_x(s_2|\pi)} = c_2 \Rightarrow P_x(s_2|\pi) = \frac{1}{1 + c_1 + c_2}.$$

We can identify probabilities in other regions in a similar manner. Region 1 will have regions 2 and 3 as its neighbors. As long as $\alpha \neq 0.5$, we know that by crossing a border between neighboring regions, either $P_\alpha(\cdot|A_2)$ or $P_\alpha(\cdot|\pi)$ changes. If it is $P_\alpha(\cdot|\pi)$, then all probabilities change; if it is $P_\alpha(\cdot|A_2)$, then only probabilities of states related to one of the events will change. By looking at regions 1 and 2, we can see that $P(s_1|\pi)$ stays the same, while other probabilities change. This implies that the subjective partition is $\pi = \{s_1, A_2\}$, where $A_2 = \{s_2, s_3\}$.

From the fact that $P_x(s_1|\pi) = P_\alpha(s_1|\pi)$ in region 1, we can recover $P_\alpha(A_2|\pi) = 1 - P_x(s_1|\pi)$; then, $P_\alpha(s_i|A_2) = \frac{P_x(s_i|\pi)}{P_\alpha(A_2|\pi)}$. Thus, all $P_\alpha(\cdot|\pi)$ and $P_\alpha(\cdot|A_2)$ in all regions can be identified. Regions 1 and 2 are from different sides of the same conditional stage boundary, meaning that $P_{\max}(\cdot)$ and $P_{\min}(\cdot)$ are the same but exchange places in order to obtain $P_\alpha(\cdot|A_2)$. Thus, we know that

$$\alpha x + (1-\alpha)y = \alpha\phi + 1 - \alpha = c_1$$
$$\alpha y + (1-\alpha)x = \alpha + (1-\alpha)\phi = c_2.$$

where x and y are vertices of the set $\mathcal{P}(A_2)$ that denote boundaries for $P(s_2|A_2)$. Note that it is a system of two equations and three unknowns. Even though a similar system of equations can be obtained from $P_\alpha(A_2|\pi)$, in total, there will be four equations and five unknowns. Thus, unique identification of the parameter α and sets of priors is not possible. However, we can achieve partial identification of parameters. The above system of equations suggests that $x + y = c_1 + c_2 = 1 + \phi > 1$. Thus, by taking into account that $0 \leq x < y \leq 1$, the biggest possible set of priors that satisfies the above system of equations is $P(s_2|A_2) \in [c_1 + c_2 - 1; 1] = [\phi; 1]$. Moreover, other possible sets of priors belong to $[\phi; 1]$.

We can use the system of equations in order to obtain partial identification of the parameter α. Note that $\alpha = \frac{c_2 - x}{y - x}$ is a monotone function of x. Because $c_2 > c_1$ (due to $\alpha > 0.5$), the function is strictly increasing in x. Thus, the smallest possible value of α is obtained when $x = \phi$ – i.e., $\alpha_{\min} = \frac{c_2 - \phi}{1 - \phi} = \alpha$, which is the true value of the parameter. Thus, the interval is $\mathcal{A} = [\alpha; 1)$.

The same idea can be applied for partial identification of the ex-ante set of priors $\mathcal{P}(\pi)$. In this example, intervals obtained from both problems for α coincide. However, generally, one needs to take into account all available equations.

4.3 Identification

Thus, the agent wants to buy a portfolio of securities and aggregates states of the world into some partition. The purpose of this section is to identify the agent's state aggregation and, when possible, priors from choices of Arrow assets and their prices. In order to do so, we generalize the above non-parametric example; however, the idea behind the method stays the same.

A central assumption for identification from choices in this paper is polytope priors that allow for the agent's choice behavior to be described by the SEU in different regions of the act space with different probabilities across regions. Moreover, this act space separation is state-aggregation-unique for the strict α-MEU stage model.

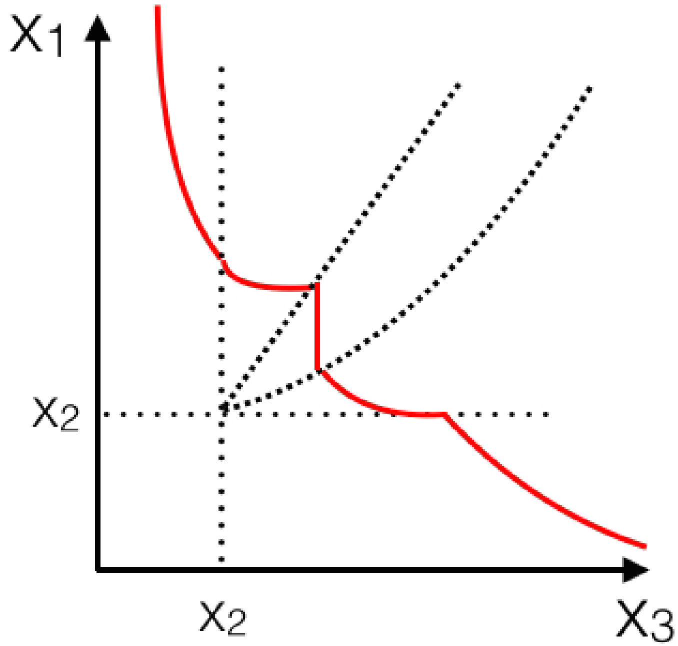


Figure 3: Typical indifference curve

Assumption 2. *Sets of priors $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$ for any $A \in \pi$ are closed convex non-empty polytopes.*

Variation of probabilities across regions creates kinks in indifference curves, as can be seen in Figure 3. However, we do not observe choices from indifference curves completely. Our data are choices of bundles of Arrow securities together with their prices. Thus, if indifference curves are not convex (as in Figure 3), then we can recover only the intersection of an indifference curve with the convex hull of its upper counter set, as shown in Figure 4.

In order to achieve convexity of indifference curves and have the optimality condition hold in each region, we assume the following:

Assumption 3. *Utility function $u(\cdot)$ is concave and differentiable.*

Only one connected set of indifferent bundles will be recovered in each observable region because indifference curves are convex there. Unfortunately, there is no guarantee that each region is represented by a set of choices. As Figure 4 shows, two middle regions are missing

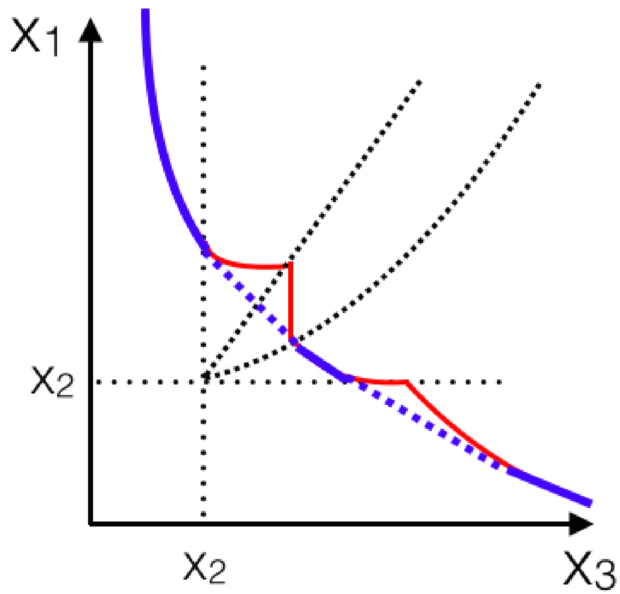


Figure 4: Intersection of IC with its convex hull

in the observations. When behavior in some region is not observed, the identification becomes impossible, which prompts the following assumption:

Assumption 4. *There exist prices and incomes such that more than one bundle is chosen in each region.*

Assumptions 1-4 help us identify the utility function up to affine transformation in each separate region. However, for the identification of the probabilities and the state aggregation, we need to connect these recovered functions with each other. In order to avoid this problem, we make another assumption:

Assumption 5. *Observed choices include payoffs from the whole compact subset X_S of the support X of $u(\cdot)$.*

Proposition 4. *If assumptions 1-5 hold, and $0 < \alpha < 1$, then the following can be uniquely identified from choice behavior:*

1. *utility $u(\cdot)$ up to affine transformation on the set X_S;*

2. *state aggregation π;*

3. *$P_x(s|\pi)$, $P_{\alpha,x}(s|A)$ and $P_{\alpha,x}(A|\pi)$ in each region.*

   In addition, α and sets of priors, $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$, can be partially identified:

4. *there exists an interval $\mathcal{A} \subset (0,1)$ such that $\alpha \in \mathcal{A}$:*

   (a) *if $\alpha > 0.5$, then $\mathcal{A} \subset (0.5, 1)$;*

   (b) *if $\alpha < 0.5$, then $\mathcal{A} \subset (0, 0.5)$;*

5. *there exist sets of priors $\widetilde{\mathcal{P}}(A)$ and $\widetilde{\mathcal{P}}(\pi)$ such that $\mathcal{P}(A) \subseteq \widetilde{\mathcal{P}}(A)$ and $\mathcal{P}(\pi) \subseteq \widetilde{\mathcal{P}}(\pi)$.*

The proof of the above proposition suggests an identification method to proceed. First, the researcher needs to determine regions of the act space in which the agent evaluates acts according to SEU with probabilities that are different across the regions. Then, utility, together with the region-specific probabilities, can be identified. And, finally, the way that probabilities change between neighboring regions suggests the state aggregation and ambiguity attitude. However, the actual parameter of ambiguity aversion α and the sets of ex-ante and conditional priors are not identified due to a non-unique representation of preferences (this problem is generally present in the α-MEU model). However, as non-parametric example shows, partial identification can be achieved. In order to obtain unique identification, some normalization conditions are required.

Note that if we apply the identification method to the MEU model, then the identified state aggregation might not be unique due to the reasons described in Section 3. However, the set of priors is identified in this case because the value of ambiguity aversion parameter is known – i.e., $\alpha = 1$. If we obtain that the state aggregation π is different from $\pi_0 = \Omega$, then it means that the state aggregation is not unique. In this case, the agent with the state aggregation π_0 and π-rectangular priors will demonstrate identical behavior. However, if the identification method provides the state aggregation π_0, then we know that it is unique, and the agent does not aggregate states.

5 Testable restrictions

This section provides a set of bilinear Afriat inequalities that allow us to test the state aggregation model with a finite dataset. Since the system of inequalities is bilinear, the Tarski-

Seidenberg algorithm can be applied to determine whether the solution exists.

An example of a dataset that could be used for testing the model is yearly individual data on car insurance deductible choices and income. The insurance plan should have at least two deductibles that represent states of the world of different occurrences. We also need to observe the whole choice set of insurance plans with all prices, and the choice set should be constant over time. Another requirement on the structure of the insurance plan must hold: the insurance price must be linear in its deductible prices – a property that is easy to verify in the data. The insurance plan with deductibles can be treated as a portfolio of Arrow assets, as discussed in Burkovskaya (2017). However, in order to recover all Arrow prices and the amount of Arrow securities, we also need to have data on individual income and losses in each state.

A dataset is a finite collection of pairs $(x^i, p^i)_{i=1}^N \in \mathbb{R}_+^\Omega \times \mathbb{R}_{++}^\Omega$, where N is a number of observations. We define a budget set as $B^i = \{x \in \mathbb{R}_+^\Omega : p^i x \leq p^i x^i\}$ and a boundary of the budget set as $\partial B^i = \{x \in \mathbb{R}_+^\Omega : p^i x = p^i x^i\}$. Also, $\mathcal{X} = \{x \in \mathbb{R}_+ : x = x_s^i \text{ for some } t, s\} \cup \{0\}$ is observed at some state consumption set together with 0; and lattice $\mathcal{L} = \mathcal{X}^\Omega$ is its product over states.

Definition 3. *A dataset $(x^i, p^i)_{i=1}^N$ is α-MEU π state aggregation rational (π-rational) if there are sets of priors $\mathcal{P}(\pi)$ and $\mathcal{P}(A)$, $A \in \pi$, a parameter α, and a continuous and increasing function $u : \mathbb{R}_+ \to \mathbb{R}$ such that for all i*

$$y \in B^i \Rightarrow V(y|\pi) \leq V(x^i|\pi).$$

Proposition 5. *A dataset is π-rational if and only if for each element of lattice $x \in \mathcal{L}$, there exist non-negative numbers $\mu_s(x)$, $\mu_{\alpha;A}(x)$, $\mu_{\alpha;s}(x)$, $q_{1;s}^A(x)$, $q_{2;s}^A(x)$, $w_{1,A}(x)$, $w_{2;A}(x)$, $0 \leq a \leq 1$, and an increasing function $\bar{u} : \mathcal{X} \to \mathbb{R}$ such that*

$$\sum_{s\in\Omega} \mu_s(x) = \sum_{A\in\pi} \mu_{\alpha;A}(x) = \sum_{s\in\Omega} \mu_{\alpha;s}(x) = 1$$

$$\sum_{s\in A} q_{1;s}^A(x) = \sum_{s\in A} q_{2;s}^A(x) = \sum_{A\in\pi} w_{1,A}(x) = \sum_{A\in\pi} w_{2,A}(x) = 1$$

$$\mu(x^i)\bar{u}(x^i) \geq \mu(x)\bar{u}(x) \text{ for all } x \in \mathcal{L} \cap B^i$$

$$\mu(x^i)\bar{u}(x^i) > \mu(x)\bar{u}(x) \textit{ for all } x \in \mathcal{L} \cap (B^i \setminus \partial B^i)$$

$$\mu_s(x) = \mu_{\alpha;A}(x)\mu_{\alpha;s}(x), s \in A, x \in \mathcal{L}$$

$$\mu_{\alpha;s}(x) = aq_{1;s}(x) + (1-a)q_{2;s}(x), x \in \mathcal{L}$$

$$\mu_{\alpha;A}(x) = aw_{1;A}(x) + (1-a)w_{2;A}(x), x \in \mathcal{L}$$

$$\sum_{s\in A} q^A_{1;s}(x)\bar{u}(x_s) \leq \sum_{s\in A} q^A_{t;s}(x')\bar{u}(x_s), \textit{ for all } x, x' \in \mathcal{L}, t = 1,2$$

$$\sum_{s\in A} q_{2;s}(x)\bar{u}(x_s) \geq \sum_{s\in A} q^A_{t;s}(x')\bar{u}(x_s), \textit{ for all } x, x' \in \mathcal{L}, t = 1,2$$

$$\sum_{A\in\pi} w_{1;A}(x)\sum_{s\in A}\mu_{\alpha;s}(x)\bar{u}(x_s) \leq \sum_{A\in\pi} w_{t;A}(x')\sum_{s\in A}\mu_{\alpha;s}(x)\bar{u}(x_s), \textit{ for all } x, x' \in \mathcal{L}, t = 1,2$$

$$\sum_{A\in\pi} w_{2;A}(x)\sum_{s\in A}\mu_{\alpha;s}(x)\bar{u}(x_s) \geq \sum_{A\in\pi} w_{t;A}(x')\sum_{s\in A}\mu_{\alpha;s}(x)\bar{u}(x_s), \textit{ for all } x, x' \in \mathcal{L}, t = 1,2.$$

For the purposes of the proof of Proposition 5, we derive conditions for the sets of priors on the lattice and apply a result from Polisson, Quah and Renou (2015), which states that if there exists an increasing utility function on the lattice, then it can be extended to a continuous increasing utility function on the whole support.

Example 4. To simplify the example as much as possible, we assume that the set of priors is the whole simplex and $\Omega = \{s_1, s_2, s_3\}$. Suppose that we are interested in testing whether the agent does not aggregate states – i.e., $\pi = \Omega$. With the whole simplex priors, the value of each bundle x such that $x_i > x_j > x_k$ will be attained by using probabilities $\mu_i = 1 - \alpha$, $\mu_j = 0$, and $\mu_k = \alpha$. Consider the agent who chooses an allocation $(3, 1, 3)$ with prices $(2, 1, 2)$. Note that allocation $(2, 2, 3)$ is also available and is strictly inside the budget set. Thus, $(3, 1, 3) \succ (2, 2, 3)$. However, $V((2, 2, 3)|\Omega) \geq V((3, 1, 3)|\Omega)$ if the utility function is increasing:

$$V((3,1,3)|\Omega) = (1-\alpha)u(3) + \alpha u(1)$$

$$V((2,2,3)|\Omega) = (1-\alpha)u(3) + \alpha u(2).$$

Thus, $(3, 1, 3)$ cannot be chosen if the agent has the whole simplex priors and does not aggregate states.

Next, consider a state aggregation $\pi = \{\{s_1, s_2\}, s_3\}$. In this case, bundles' values are

$$V((3,1,3)|\pi) = (1-\alpha)u(3) + \alpha\left((1-\alpha)u(3) + \alpha u(1)\right)$$

$$V((2,2,3)|\pi) = (1-\alpha)u(3) + \alpha u(2).$$

Thus, $(3,1,3) \succ (2,2,3)$ if and only if $(1-\alpha)u(3) + \alpha u(1) > u(2)$, which is generally possible with some increasing function $u(\cdot)$. However, note that $u(3) - u(2) > \frac{\alpha}{1-\alpha}(u(2) - u(1))$. The last expression implies that if the agent is ambiguity-averse, then her utility function is not concave. Thus, if we require both ambiguity and risk aversion, then this data point rejects the state aggregation π with the whole simplex priors.

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A Axiomatization

A.1 Preliminaries

We denote for all $f, g \in \mathcal{F}, A \in \Sigma$, fAg an act: $fAg(s) = f(s)$ if $s \in A$, and $fAg(s) = g(s)$ if $s \notin A$. The mixture of acts is defined statewise. We will abuse notation and define X as a set of constant acts in what follows below.

acts in $\mathcal{F}$ for any partition $\pi \in \mathcal{M}$.

A.2 Axioms and representation

The purpose of this section is to provide axioms of preference relation $\succeq$ between acts f over $\mathcal{F}$ that can be represented by the model of state aggregation, as described in Section 2. We take as a primitive a preference relation between acts. After that, we induce conditional preferences whenever we can guarantee their completeness.

First, we assume that $\succeq$ satisfies the classical axioms of Gilboa and Schmeidler (1989):

Axiom 1 (Invariant Biseparable Preferences - IBP)**.** *For all $f, g, h \in \mathcal{F}$ and $x \in X$: (i) $\succeq$ is complete and transitive; (ii) if $\lambda \in (0,1]$: $f \succeq g \Leftrightarrow \lambda f + (1-\lambda)x \succeq \lambda g + (1-\lambda)x$; (iii) if $f \succ g$, and $g \succ h$, then there exist $\lambda, \mu \in (0,1)$ such that $\lambda f + (1-\lambda)h \succ g$ and $g \succ \mu f + (1-\mu)h$; (iv) if $f(s) \succeq g(s)$ for all $s \in \Omega$, then $f \succeq g$; and (v) $\succeq$ is not degenerate. i.e. there is no such state s: $f(s) \succ g(s)$ and $fsh \sim gsh$.*

In order to introduce state aggregation and obtain complete conditional preferences, we define a concept of an aggregating event:

Definition 4. *Event A is called aggregating if for any $f, g, h, h' \in \mathcal{F}$: $fAh \succeq gAh \Leftrightarrow fAh' \succeq gAh'$.*

An aggregating event satisfies a property similar to Savage's Sure Thing Principle. The independence axiom does not hold in this model, so it is not equivalent to the existence of a unique probability. The aggregating event implies that the value of the act at the event is the only aspect that matters in act evaluation, and not each state value separately.

Denote a set of all aggregating events $\mathcal{A}$. Note that $\{\Omega\}$ trivially belongs to it, and each separate state is in there due to monotonicity. Now, for all aggregating events A, we define conditional preferences $\succeq_A$:

Axiom 2 (Conditional Preferences - CP)**.** *For all $f, g, h \in \mathcal{F}$: $f \succeq_A g \Leftrightarrow fAh \succeq gAh$.*

Note that conditional preferences $\succeq_A$ are complete and satisfy all analogous axioms for IBP.

In order to obtain the $\alpha-$MEU stage preferences, we follow Ghirardato, Maccheroni, and Marinacci (2004) to define unambiguous preferences $\succeq_A^*$:

Definition 5. *For any $A \in \mathcal{A}$ and acts $f, g, h \in \mathcal{F}$, f is unambiguously preferred to g given event A, $f \succeq_A^* g$, if $\lambda f + (1-\lambda)h \succeq_A \lambda g + (1-\lambda)h$ for all $\lambda \in (0, 1]$.*

For each possible partition π, we define a set of π-measurable acts $\mathcal{F}_\pi$. Then, the unambiguous π-measurable preference relation is $\succeq_\pi^*$:

Definition 6. *For any acts $f, g, h \in \mathcal{F}_\pi$, f is unambiguously preferred to g given π, $f \succeq_\pi^* g$, if $\lambda f + (1-\lambda)h \succeq \lambda g + (1-\lambda)h$ for all $\lambda \in (0, 1]$.*

We now define "possible" certainty equivalence sets for act f on event A:

$$C_A(f) = \{x \in X : \forall y \in X : \text{ if } y \succeq_A^* f, \text{ then } y \succeq_A^* x, \text{ and if } y \preceq_A^* f, \text{ then } y \preceq_A^* x\}$$

Similarly, the "possible" certainty equivalence set for π-measurable act $f \in \mathcal{F}_\pi$ is:

$$C_\pi(f) = \{x \in X : \forall y \in X : \text{ if } y \succeq_\pi^* f \text{ then } y \succeq_\pi^* x, \text{ and if } y \preceq_\pi^* f \text{ then } y \preceq_\pi^* x\}.$$

The above sets contain all constant acts with which unambiguous preferences for f are not defined. Now we adapt Axiom 7 from Ghirardato, Maccheroni, and Marinacci (2004) to fix parameters of ambiguity aversion at each stage for all acts.

Axiom 3 (α-MEU)**.** *For all acts $f, g, h, h' \in \mathcal{F}$, there exist a partition π of Ω such that $\pi \subset \mathcal{A}$, and aggregating events $A, B \in \pi$: (i) If $C_A(f) = C_A(g)$, then $f \sim_A g$; (ii) if $C_A(f) = C_B(g)$, then there exists $x \in X$: $f \sim_A x$ and $g \sim_B x$; and (iii) if $C_\pi(f) = C_A(g)$, then there exists $x \in X$: $f \sim x$ and $g \sim_A x$.*

Theorem 6. *(Representation Theorem) A binary relation $\succeq$ satisfies axioms 1-3 if and only if there exist nonempty weak compact and convex sets $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$ for any event $A \in \pi$ of probabilities on Σ, a unique up to affine transformation nonconstant continuous monotone function $u : X \to \mathbb{R}$, and $\alpha \in [0, 1]$ such that $\succeq_A$ is represented by the unique preference functional $V(\cdot|A) : \mathcal{F} \to \mathbb{R}$, and $\succeq$ is represented by unique $V(\cdot|\pi) : \mathcal{F} \to \mathbb{R}$ such that agent's behavior exhibits α-MEU SAR, and $u(\cdot), V(\cdot|\cdot), \mathcal{P}(A)$ and $\mathcal{P}(\pi)$ represent $\succeq_A^*$ and $\succeq_\pi^*$, as defined above.*

Moreover, for each A and π, $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$ are unique, and α is unique if $\mathcal{P}(\pi)$ is not a singleton, and there exist $A \in \pi$ such that $\mathcal{P}(A)$ is not a singleton, too.

Proof. First, due to Proposition 19 from Ghirardato, Maccheroni, Marinacci (2004), axioms 1, 2, and 3(i) guarantee unique $V(\cdot|A), \mathcal{P}(A), \beta(A)$ and a unique up to affine transformation $u(\cdot)$ that represent $\succeq_A^*$. Moreover,

$$V(f|A) = \beta(A) \min_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(f(s))P(s|A) + (1-\beta(A)) \max_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(f(s))P(s|A).$$

We now show that $\beta(A) = \beta$ for all $A \in \mathcal{A}$. Suppose that $C_A(f) = C_B(g) = [x_1, x_2]$. Then, by axiom 3(ii), $f \sim_A \beta(A)x_1 + (1-\beta(A))x_2 \sim_A x$; $g \sim_B \beta(B)x_1 + (1-\beta(B))x_2 \sim_B x$. The last implies that $\beta(A) = \beta(B) = \beta$.

Next, we demonstrate that monotonicity with respect to events in π holds for the preferences $\succeq$. We need to show that if $f \succeq_A g$ for all $A \in \pi$, then $f \succeq g$. Suppose that $\pi = \{A_1, A_2, ..., A_n\}$. Consider a constant act $x_f^{A_i}$ such that $f \sim_{A_i} x_f^{A_i}$ and $fA_ih \sim x_f^{A_i}A_ih$. Also, notice that

$$f \sim x_f^{A_1}A_1f \sim x_f^{A_1}A_1x_f^{A_2}A_2f \sim ... \sim x_f^{A_1}A_1x_f^{A_2}A_2...x_f^{A_n}A_n.$$

The same will hold for act g, by analogy. However, if $f \succeq_{A_i} g$, then $x_f^{A_i} \succeq_{A_i} x_g^{A_i}$, meaning that x_f has a value higher than or equal x_g at each state. Thus, by monotonicity

$$x_f^{A_1}A_1x_f^{A_2}A_2...x_f^{A_n}A_n \succeq x_g^{A_1}A_1x_g^{A_2}A_2...x_g^{A_n}A_n. \Rightarrow f \succeq g$$

Now we demonstrate that axiom 3(iii) implies that if $C_\pi(f) = C_\pi(g)$, then $f \sim g$. Suppose that $C_\pi(f) = C_\pi(g)$; then, we take an act h such that there exists an event $A \in \pi$: $C_\pi(f) = C_\pi(g) = C_A(h)$. Then, there exists $x \in X$: $f \sim x$ and $g \sim x$, so $f \sim g$.

Next, we apply Proposition 19 from Ghirardato, Maccheroni, Marinacci (2004) again. However, we will treat events from π as states. We obtain unique $V(\cdot|\pi), \mathcal{P}(\pi), \alpha$ and unique up to affine transformation $I(\cdot)$ that represent $\succeq^*$, and the representation

$$V(f|\pi) = \alpha \min_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} I(f|A)P(A|\pi) + (1-\alpha) \max_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} I(f|A)P(A|\pi).$$

Note that $I(x|A)$ and $V(x|A)$ represent the same preferences, and this means that they are monotone transformations of each other. However, they have to be equal on constant acts, so they coincide. Also, notice that $\alpha = \beta$ follows straight from axiom 3(iii). $\square$

The uniqueness of representation is related to specific partition π. Also parameters and priors are unique in their representation of unambiguous preferences. However, Ghirardato, Maccheroni, and Marinacci (2004) find that some other set of priors and parameters representing the same preferences might exist, but they will not be related to the unambiguous preferences $\succeq^*$. Moreover, the proposed set of priors will be bigger, and the parameter of ambiguity aversion will be closer to $\frac{1}{2}$.

B Appendix

Suppose that there are two different partitions, π and π'. Denote a set $\tilde{\pi} = \{C \in \Sigma : \exists A \in \pi, B \in \pi' : C = A \cap B\}$.

Lemma 1. *If $\pi, \pi' \in \mathcal{A}$ and $C \in \tilde{\pi}$, then $C \in \mathcal{A}$.*

$f, g, h \in \mathcal{F}$ and $x \in X$: $fCx \succeq_A gCx \Leftrightarrow fCh \succeq_A gCh$ due to the same preferences.

Proof. Take events $A \in \pi$ and $B \in \pi'$ such that $C = A \cap B$. Then, $fCx \succeq_A gCx \Leftrightarrow fCx(A \setminus C)h \succeq gCx(A \setminus C)h$. The last relation is equivalent to $fCh(B \setminus C)x(A \setminus C)h \succeq gCh(B \setminus C)x(A \setminus C)h \Leftrightarrow fCh \succeq_B gCh$.

Now, by providing the same argument from event B back to A, one can easily obtain that $fCh \succeq_A gCh$. $\square$

Lemma 2. *Suppose that $f(C_1 \cup C_2)x_f \sim_B x_f$; $fC_1x_1 \sim_{A_1} x_1$ and $fC_2x_2 \sim_{A_2} x_2$, $x_1 \neq x_2$ and $A_i \cap B = C_i$. Then, there exists i (1 or 2) such that $x_i = x_f$ if and only if $\alpha = 0, 1$ and for any $s \in B$: $P(s|B) = 0$ is in the set of priors $\mathcal{P}(B)$.*

Proof. $x_1 C_1 x_2 C_2 x_f \sim_B x_f$. Then,

$$V(x_1C_1x_2C_2x_f) - u(x_f) = \alpha \min_p \left(P(C_1|B)(u(x_1) - u(x_f)) + P(C_2|B)(u(x_2) - u(x_f))\right) +$$
$$+(1-\alpha)\max_p \left(P(C_1|B)(u(x_1) - u(x_f)) + P(C_2|B)(u(x_2) - u(x_f))\right) = 0.$$

Suppose that $x_j = x_f$, and, for simplicity, $x_i > x_f$. Then,

$$V(x_jC_jx_iC_ix_f) - u(x_f) = (u(x_i) - u(x_f))(\alpha P_{\min} + (1-\alpha)P_{\max}) = 0,$$

which implies that either $\alpha = 0$ and $P_{\max}(C_i|B) = 0$ or $\alpha = 1$ and $P_{\min}(C_i|B) = 0$. Thus, $P(C_i|B) = 0$ is available in the set of priors.

Now, suppose that for each state s: $P(s|B) = 0$ is available. Suppose that $x_j > x_f > x_i$:

$$(u(x_j) - u(x_f))(\alpha P^j_{\min} + (1-\alpha)P^j_{\max}) + (u(x_i) - u(x_f))(\alpha P^i_{\min} + (1-\alpha)P^i_{\max}) = 0.$$

Then, $P^j_{\min} = P^i_{\max} = 0$ and $P^i_{\min} = P^j_{\max} = 1$. This implies that $u(x_f) = \alpha u(x_i) + (1-\alpha)u(x_j)$. Note that if $\alpha \neq 0, 1$, then there is no such i that $x_f = x_i$. □

Lemma 3. *If there exists $C \notin \pi$, but $C \in \mathcal{A}$, and there exists $A \in \pi$: $C \subset A$ and C consists of two or more states, then one of the following must be true:*

1. *$P(C|A) = \sum_{s\in C} P(s|A)$ is a constant;*

2. *For all $s \in C$: $P(s|C) = \frac{P(s|A)}{\sum_{s\in C} P(s|A)}$ is a constant;*

3. *$\alpha = 0, 1$ and either $\exists s \in C$: $P(s|A) = 0$ is in the set of priors $\mathcal{P}(A)$ or $\mathcal{P}(A)$ is C-rectangular.*

Proof. We define constant act x_f: $fCx_f \sim_A x_f$. This implies that $fCh \sim_A x_fCh$ for any act h. To simplify calculations, instead of h, we take some constant act y. Thus, $V(fCy|A) = V(x_fCy|A)$, where values are

$$V(fCy|A) = \alpha \min_{p(\cdot|A)} \left(\sum_{s\in C} p(s|A)u(f(s)) + \sum_{s\in A\setminus C} p(s|A)u(y)\right) + (1-\alpha)\max_{p(\cdot|A)} \left(\sum_{s\in C} p(s|A)u(f(s)) + \sum_{s\in A\setminus C} p(s|A)u(y)\right)$$

$$V(x_f C y|A) = \alpha \min_{p(\cdot|A)} \left(P(C|A)u(x_f) + \sum_{s \in A \setminus C} p(s|A)u(y) \right) + (1-\alpha) \max_{p(\cdot|A)} \left(P(C|A)u(x_f) + \sum_{s \in A \setminus C} p(s|A)u(y) \right).$$

Now, by subtracting $u(x_f)$ from both expressions and setting them equal to each other, one can obtain the following:

$$\alpha \min_{p(\cdot|A)} \left(\sum_{s \in C} p(s|A)(u(f(s)) - u(x_f)) + (1 - P(C|A))(u(y) - u(x_f)) \right) +$$
$$+ (1-\alpha) \max_{p(\cdot|A)} \left(\sum_{s \in C} p(s|A)(u(f(s)) - u(x_f)) + (1 - P(C|A))(u(y) - u(x_f)) \right) =$$
$$= \alpha \min_{p(\cdot|A)} (1 - P(C|A))(u(y) - u(x_f)) + (1-\alpha) \max_{p(\cdot|A)} (1 - P(C|A))(u(y) - u(x_f)).$$

The above expression must hold for any acts y and f such that $fCx_f \sim_A x_f$.

If the C-related part inside min and max is always 0, then one of the following is true: (1) for any $s \in C$: $P(s|C) = const$; (2) $\alpha = 0, 1$ and there always exists s such that $P(s|A) = 0$ is available in the set of priors that, by Lemma 2, would imply that there is always $u(f(s)) = u(x_f)$.

If the C-related part is not always 0, then the set of priors has to satisfy some kind of separability between events C and $A \backslash C$. It means one of the following must hold: (1) $P(C|A) = const$; (2) $\alpha = 0, 1$ and $\mathcal{P}(A)$ is C-rectangular. ☐

Lemma 4. *Suppose that there are events $A_1, A_2 \in \pi$ and $B \in \pi'$, $A_i \cap B = C_i$ and there is i such that $A_i \neq C_i$; then, one of the following holds:*

1. *$P(A_i)$ is a constant;*
2. *$\frac{P(A_i)}{P(A_1)+P(A_2)}$ is a constant;*
3. *$\alpha = 0, 1$ and there is state $s \in C_i$ such that $P(s|A_i) = 0$ is available in the set of priors $\mathcal{P}(A_i)$ or $P(A_i) = 0$ is available in the set of priors $\mathcal{P}(\pi)$.*

Proof. Define $f(C_1 \cup C_2)x_f \sim_\pi x_f$ and $fC_i x_i \sim_{A_i} x_i$. Note that $f(C_1 \cup C_2)h \sim_\pi x_f(C_1 \cup C_2)h$. Now, instead of act h, we take an act such that it gives x_i on $A_i \setminus C_i$ and some constant act y

everywhere else. Then,

$$x_fC_1x_1(A_1\setminus C_1)x_fC_2x_2(A_2\setminus C_2)y\sim_\pi f(C_1\cup C_2)x_1(A_1\setminus C_1)x_2(A_2\setminus C_2)y\sim_\pi x_1A_1x_2A_2y.$$

We denote act $g=x_fC_1x_1(A_1\setminus C_1)x_fC_2x_2(A_2\setminus C_2)y$. Note that

$$V(g)-u(x_f)=\alpha\min_{P(\cdot|\pi)}\left(\sum_{A_i}P(A_i|\pi)(V(x_fC_ix_i|A_i)-u(x_f))+(1-\sum_{A_i}P(A_i))(u(y)-u(x_f))\right)+$$
$$+(1-\alpha)\max_{P(\cdot|\pi)}\left(\sum_{A_i}P(A_i|\pi)(V(x_fC_ix_i|A_i)-u(x_f))+(1-\sum_{A_i}P(A_i))(u(y)-u(x_f))\right),$$

where

$$V(x_fC_ix_i|A_i)-u(x_f)=\alpha\min_{P(\cdot|A_i)}\left(\sum_{s\in A_i\setminus C_i}P(s|A_i)(u(x_i)-u(x_f))\right)+(1-\alpha)\max_{P(\cdot|A_i)}\left(\sum_{s\in A_i\setminus C_i}P(s|A_i)(u(x_i)-u(x_f))\right)=$$
$$=(u(x_i)-u(x_f))\,(1-(\alpha P_{\min}(C_i|A_i)+(1-\alpha)P_{\max}(C_i|A_i)))=(u(x_i)-u(x_f))\,(1-P_\alpha(C_i|A_i)).$$

$V(x_fC_ix_i|A_i)-u(x_f)$ can be 0 either if $C_i=A_i$ or if $x_i=x_f$. By Lemma 2, the last implies that $\alpha=0,1$, and there is $s\in C_i$ such that $P(s|A_i)=0$ is in the set of priors $\mathcal{P}(A_i)$. Thus,

$$V(g)-u(x_f)=\alpha\min_{P(\cdot|\pi)}\left(\sum_{A_i}(u(x_i)-u(x_f))\,(1-P_\alpha(C_i|A_i))P(A_i)+(1-\sum_{A_i}P(A_i))(u(y)-u(x_f))\right)+$$
$$+(1-\alpha)\max_{P(\cdot|\pi)}\left(\sum_{A_i}(u(x_i)-u(x_f))\,(1-P_\alpha(C_i|A_i))P(A_i)+(1-\sum_{A_i}P(A_i))(u(y)-u(x_f))\right)=$$
$$=\sum_{A_i}(u(x_i)-u(x_f))\,(1-P_\alpha(C_i|A_i))\tilde{P}_\alpha(A_i)+(1-\sum_{A_i}\tilde{P}_\alpha(A_i))(u(y)-u(x_f)).$$

We now compare it with the value of $x_1A_1x_2A_2y$:

$$V(x_1A_1x_2A_2y) - u(x_f) = \alpha \min_{P(\cdot|\pi)} \left(\sum_{A_i} P(A_i)(u(x_i) - u(x_f)) + (1 - \sum_{A_i} P(A_i))(u(y) - u(x_f)) \right) +$$
$$+(1-\alpha) \max_{P(\cdot|\pi)} \left(\sum_{A_i} P(A_i)(u(x_i) - u(x_f)) + (1 - \sum_{A_i} P(A_i))(u(y) - u(x_f)) \right) =$$
$$= \sum_{A_i} P_\alpha(A_i)(u(x_i) - u(x_f)) + (1 - \sum_{A_i} P_\alpha(A_i))(u(y) - u(x_f)).$$

The above expressions will trivially coincide if A_i-related parts are always 0. This is possible in one of the following situations: (1) $\frac{P(A_i)}{P(A_1)+P(A_2)}$ is a constant and $C_i = A_i$; (2) $P(A_i)$ is constant; (3) there exists some i: $x_i = x_f$, and $P(A_j) = 0$ is in the set of priors $\mathcal{P}(\pi)$, which imply that $\alpha = 0, 1$, and $s_i \in A_i$ such that $P(s_i|A_i) = 0$ is in the set of priors.

Now, suppose that A_i-related parts are not always 0. The expressions must be equal to each other for any value of y, which implies that $\tilde{P}_\alpha(A_1) + \tilde{P}_\alpha(A_2) = P_\alpha(A_1) + P_\alpha(A_2)$. Note that $u(x_f) = wu(x_1) + (1-w)u(x_2)$, where $w \in (0,1)$. By substituting it into both expressions and setting them equal to each other, one can obtain:

$$w = \frac{P_\alpha(A_1) - (1 - P_\alpha(C_1|A_1))\tilde{P}_\alpha(A_1)}{P_\alpha(A_1) - (1 - P_\alpha(C_1|A_1))\tilde{P}_\alpha(A_1) + P_\alpha(A_2) - (1 - P_\alpha(C_2|A_2))\tilde{P}_\alpha(A_2)}.$$

The above ratio must be constant for any value of y, while $P_\alpha(C_i|A_i)$ does not change, depending on y. This implies that $\frac{P_\alpha(A_1)-(1-P_\alpha(C_1|A_1))\tilde{P}_\alpha(A_1)}{P_\alpha(A_2)-(1-P_\alpha(C_2|A_2))\tilde{P}_\alpha(A_2)} = c$, where c is a constant. If $P_\alpha(A_2)$ and $\tilde{P}_\alpha(A_2)$ are not equal to 0 (this would imply that 0 is in the set of priors for A_i), then the last statement can be rewritten as follows

$$\left(\frac{P_\alpha(A_1)}{P_\alpha(A_2)} - c \right) \frac{P_\alpha(A_2)}{\tilde{P}_\alpha(A_2)} = (1 - P_\alpha(C_1|A_1)) \frac{\tilde{P}_\alpha(A_1)}{\tilde{P}_\alpha(A_2)} - c(1 - P_\alpha(C_2|A_2)).$$

On the other hand, $\tilde{P}_\alpha(A_1) + \tilde{P}_\alpha(A_2) = P_\alpha(A_1) + P_\alpha(A_2)$, which can be rewritten as

$$\left(\frac{P_\alpha(A_1)}{P_\alpha(A_2)} + 1 \right) \frac{P_\alpha(A_2)}{\tilde{P}_\alpha(A_2)} = \frac{\tilde{P}_\alpha(A_1)}{\tilde{P}_\alpha(A_2)} + 1.$$

By dividing the first expression by the second one, we get:

$$\frac{\frac{P_\alpha(A_1)}{P_\alpha(A_2)} - c}{\frac{P_\alpha(A_1)}{P_\alpha(A_2)} + 1} = \frac{(1 - P_\alpha(C_1|A_1))\frac{\tilde{P}_\alpha(A_1)}{\tilde{P}_\alpha(A_2)} - c(1 - P_\alpha(C_2|A_2))}{\frac{\tilde{P}_\alpha(A_1)}{\tilde{P}_\alpha(A_2)} + 1}.$$

The above expression depends only on the ratios $\frac{P_\alpha(A_1)}{P_\alpha(A_2)}$ and $\frac{\tilde{P}_\alpha(A_1)}{\tilde{P}_\alpha(A_2)}$ and not on the actual values. It is possible in one of the following situations:

1. $c = 0$ implies $P_\alpha(C_1|A_1) = 0$ and $P_\alpha(A_i) = \tilde{P}_\alpha(A_i)$. It means that either $\alpha = 0, 1$ and $P(C_1|A_1) = 0$ is available in $\mathcal{P}(A_1)$ or $P(C_1|A_1) = 0$.

2. $c \neq 0$ implies that $P_\alpha(C_i|A_i) = 0$.

□

Proof of Proposition 1. Suppose that $\pi \neq \pi'$. Then, in one of the partitions (suppose π), there exists event A such that it intersects with at least two events from another partition (π'). Then, only two options are possible: either $A = \cup_i B_i$ or $A \neq \cup_i B_i$ for some $B_i \in \pi'$.

First, suppose that $A = \cup_i B_i$. Now, in order to use Lemma 3, replace s_i with B_i, C with A, and A with π. Then, either $P(A|\pi')$ or $P(B_i|A)$ is a constant. Unless $A = \Omega$, the last two implications suggest that priors are not full-dimensional. If at least one of B_i consists of two or more states, then, again, by Lemma 3, either $P(B_i|A)$ is a constant or for $s \in B_i$ $P(s|B_i)$ is a constant. Thus, the only possibility left is when $A = \Omega$ and all $B_i = s_i$, which means that we are comparing $\pi_0 = \{\Omega\}$ and $\pi^* = \Omega$, that we consider to be the same in this paper.

Finally, suppose that $A \neq \cup_i B_i$. Then, by Lemma 4, we know that either $P(B_i)$ or $\frac{P(B_i)}{\sum P(B_i)}$ is a constant, which contradicts full-dimensionality of priors. □

Proof of Proposition 2. Suppose that preferences are represented by some π and $\alpha = 1$ with priors $\mathcal{P}(\pi)$ and $\mathcal{P}(A)$ for each $A \in \pi$. Then, the value functional is

$$V(f|\pi) = \min_{q \in \mathcal{P}(\pi)} \sum_{A \in \pi} q(A) \left(\min_{p \in \mathcal{P}(A)} pu(f) \right) = \min_{p \in \mathcal{P}(\pi_0)} pu(f),$$

where $\mathcal{P}(\pi_0)(s) = \mathcal{P}(\pi)(A) \cdot \mathcal{P}(A)(s)$ if $s \in A$ – i.e., $\mathcal{P}(\pi_0)$ is π-rectangular. □

Proof of Proposition 3. Preferences are represented by some partition π, so by Proposition 2, the value functional can be rewritten through the trivial partition π_0 with π-rectangular priors $\tilde{\mathcal{P}}(\pi_0)$. On the other hand, preferences can be represented by the trivial partition with non π-rectangular priors $\mathcal{P}(\pi_0)$. Due to the uniqueness of representation of MEU preferences, priors must coincide. Thus, $\pi = \pi_0$.

□

Lemma 5. *For any act $x \in \mathcal{F}$ and partition π, the value $V(x|\pi)$ can be written as follows:*

$$V(x|\pi) = \sum_s P_x(s|\pi)u(x(s)).$$

Proof. From the representation theorem:

$$V(x|A) = \alpha \min_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(x(s))P(s|A) + (1-\alpha) \max_{P(\cdot|A)\in\mathcal{P}(A)} \sum_{s\in A} u(x(s))P(s|A) =$$
$$= \sum_{s\in A} \left(\alpha P_{\min x}(s|A) + (1-\alpha)P_{\max x}(s|A)\right) u(x(s)) = \sum_{s\in A} P_{\alpha;x}(s|A)u(x(s))$$

$$V(x|\pi) = \alpha \min_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} V(x|A)P(A|\pi) + (1-\alpha) \max_{P\in\mathcal{P}(\pi)} \sum_{A\in\pi} V(x|A)P(A|\pi) =$$
$$= \sum_{A\in\pi} \left(\alpha P_{\min x}(A|\pi) + (1-\alpha)P_{\max x}(A|\pi)\right) V(x|A) = \sum_{A\in\pi} P_{\alpha;x}(A|\pi)V(x|A) =$$
$$= \sum_{A\in\pi} P_{\alpha;x}(A|\pi) \sum_{s\in A} P_{\alpha;x}(s|A)u(x(s)) = \sum_s P_{\alpha;x}(A|\pi)P_{\alpha;x}(s|A)u(x(s)),$$

where $s \in A \in \pi$. □

Lemma 6. *If assumptions 1 and 2 hold, then the act space $\mathcal{F}$ can be split into a finite number of regions with $P_x(s|\mathcal{A})$ constant inside each of them.*

Proof. Consider the conditional stage when event A is observed. If event A has only one state s, then $\mathcal{P}(A)$ is a singleton, and $P(s|A) = 1$ for any act. Thus, the act space is not being split by this event.

Now, suppose that event A consists of two or more states. Remember that $P_{\alpha;x}(s|A) = \alpha P_{\min x}(s|A) + (1-\alpha) P_{\max x}(s|A)$, where

$$P_{\min x}(\cdot|A) = \underset{P(\cdot|A)\in\mathcal{P}(A)}{\operatorname{argmin}} \sum_{s\in A} P(s|A)u(x(s))$$
$$P_{\max x}(\cdot|A) = \underset{P(\cdot|A)\in\mathcal{P}(A)}{\operatorname{argmax}} \sum_{s\in A} P(s|A)u(x(s)).$$

Since $\mathcal{P}(A)$ is a closed convex bounded polytope, $P_{\min x}(s|A)$ and $P_{\max x}(s|A)$ are solutions to the linear-programming problem. As a result, they lie on the boundary of the set $\mathcal{P}(A)$ and depend on the direction of the vector $(u(x(s_1)), ..., u(x(s_k)))$, where $A = \{s_1, ..., s_k\}$. Moreover, when the solution is at the corner of $\mathcal{P}(A)$, it will be the same for all vectors $(u(x(s_1)), ..., u(x(s_k)))$ in between two hyperplanes that created the corner in the conditional set of priors. Because the number of corners in a closed convex polytope is finite, there is a finite number of regions in the act space where $P_{\min x}(\cdot|A)$ is constant. The same is also true for $P_{\max x}(\cdot|A)$, implying it for $P_{\alpha;x}(\cdot|A)$.

The same argument can be applied to the ex-ante stage and $P_{\alpha;x}(\cdot|\mathcal{A})$. □

Lemma 7. *If assumptions 1 and 2 hold, $0 < \alpha < 1$, and two choice sets are generated by preferences represented by partitions $\pi_1 \neq \pi_2$, then the obtained sets of regions in the act space differ.*

Proof. At the conditional stage given event $A = \{s_1, ..., s_n\}$, the agent's problem of finding probabilities can be reformulated as follows:

$$\sum_{i=1}^{n-1} (u(x(s_i)) - u(x(s_n)))p_i \to \max / \min$$
$$\text{s.t. } p \in \mathcal{P}(A).$$

The direction of utility growth is defined by the vector $\mathbf{u} = (u(x(s_1)) - u(x(s_n)), ..., u(x(s_{n-1})) - u(x(s_n)))'$.

$\mathcal{P}(A)$ is a polytope, so it is constructed by the intersection of several hyperplanes. Each hyperplane has two (opposite) normal unit vectors. Thus, $\mathcal{P}(A)$ creates a set of unit vectors

$W(\mathcal{P}(A))$ perpendicular to the hyperplanes of $\mathcal{P}(A)$. $W(\mathcal{P}(A))$ splits the vector space it belongs to into different regions. When $\mathbf{u}$ lies in between $n-1$ neighbor vectors from $W(\mathcal{P}(A))$, the solution from $\mathcal{P}(A)$ is the same for all acts with vector $\mathbf{u}$ in this region. Thus, the act space gets separated by boundaries where $\mathbf{u}$ has the same direction as some $v = (v_1, ...v_{n-1})' \in W(\mathcal{P}(A))$. Then, the boundary can be described by the following system of equations:

$$\begin{aligned} u(x(s_1)) - u(x(s_n)) &= \lambda v_1 \\ &... \\ u(x(s_{n-1})) - u(x(s_n)) &= \lambda v_{n-1} \\ \lambda &= \sqrt{\sum_{i=1}^{n-1} (u(x(s_i)) - u(x(s_n)))^2}. \end{aligned}$$

Due to the symmetry of $W(\mathcal{P}(A))$, for each $v \in W(\mathcal{P}(A))$, all conditional stage boundaries will be defined by the following condition:

$$u(x(s_i)) = \frac{v_i}{v_j} u(x(s_j)) + \left(1 - \frac{v_i}{v_j}\right) u(x(s_n)) \text{ for any } i = \overline{1, n-2},$$

where s_j is a state for which $v_j \neq 0$ for the given vector (such state always exists). Thus, notice that the conditional stage boundaries depend on payoffs in the states in event A only.

One potential problem might arise here. Suppose that there is another partition that includes two events B and C such that $A = B \cup C$. Is it possible that the conditional boundaries produced by A are the same as those produced by B and C? In order for this to happen, each boundary must split states into two groups. It is possible only if $u(x(s_i)) - u(x(s_n)) = \lambda v$ for all i such that $s_i \in B$, and $u(x(s_j)) - u(x(s_n)) = 0$ for all j such that $s_j \in C$. This must hold for all conditional boundaries of event A. However, it implies that either $\mathcal{P}(A)$ is not a polytope or it is not full-dimensional. Thus, under assumptions 1 and 2, the problem does not arise.

By applying the same procedure to the ex-ante stage, the ex-ante boundary can be obtained, too:

$$V(x|A_i) = \frac{w_i}{w_j} V(x|A_j) + \left(1 - \frac{w_i}{w_j}\right) V(x|A_k) \text{ for any } i = \overline{1, k-1},$$

where $w \in W(\mathcal{P}(\pi))$, $\pi = \{A_1, ..., A_k\}$ and A_j is an event for which $w_j \neq 0$. Note that

$$V(x|A) = \sum_{s \in A} P_{\alpha;x}(s|A)u(x(s)),$$

and it implies that the ex-ante boundary depends on all states in Ω.

Thus, if all boundaries are the same for two different people, it means that the conditional stage boundaries must coincide. This occurs because the conditional stage boundary depends only on payoffs in the states of the specific event, while the ex-ante stage boundary includes all states. They can coincide only between π_0 and π^*. Now, note that the conditional stage boundaries imply the partition. Thus, both agents have the same subjective partition. □

Proof of Proposition 4. Assumption 4 guarantees that we observe each region. Regions are separated from each other either by corner points or by holes. Corner points are those bundles that are chosen under the range of price ratios. Holes are bundles that are never chosen under any combination of price ratios and income. Inside some region R, the value functional is SEU with probabilities $P_x(s|\pi)$ of each state s. Then, inside the region the optimality condition holds:

$$\frac{p_j}{p_i} = \frac{P_x(s_j|\pi)}{P_x(s_i|\pi)} \frac{u'(x(s_j))}{u'(x(s_i))}.$$

We fix some value of $x(s_i) = \bar{x}$ and take a look at the values of $x(s_j)$ such that the bundle is still in region R. Then, observed $x(s_j)$ will belong to some closed interval $[a(\bar{x}), b(\bar{x})]$ ($b(\bar{x})$ might be $+\infty$). Now, note that for any $x_1, x_2 \in [a(\bar{x}), b(\bar{x})]$, there exist prices p_i^1, p_i^2, p_j^1 and p_j^2 such that:

$$\frac{p_j^1}{p_i^1} = \frac{P_f(s_j|\pi)}{P_f(s_i|\pi)} \frac{u'(x_1)}{u'(\bar{x})}$$

$$\frac{p_j^2}{p_i^2} = \frac{P_f(s_j|\pi)}{P_f(s_i|\pi)} \frac{u'(x_2)}{u'(\bar{x})}$$

$$\Rightarrow \frac{u'(x_1)}{u'(x_2)} = \frac{p_j^1}{p_i^1} \frac{p_i^2}{p_j^2}.$$

Thus, the ratio $\frac{u'(x_1)}{u'(x_2)}$ is identified in the interval $[a(\bar{x}), b(\bar{x})]$. By analogy, such a ratio is identified in all intervals that are observed in different states.

If $\frac{u'(x_1)}{u'(x_2)}$ is identified in two different intervals $[a_1, b_1]$ and $[a_2, b_2]$ and $a_1 \leq a_2 \leq b_1 \leq b_2$, then it is also identified in $[a_1, b_2]$: for any $x_1 \in [a_1, b_1]$, $x_2 \in [a_2, b_2]$ and $z \in [a_2, b_1]$, the following holds

$$\frac{u'(x_1)}{u'(x_2)} = \frac{u'(x_1)/u'(z)}{u'(x_2)/u'(z)}.$$

The only potential problem with identification of $\frac{u'(x_1)}{u'(x_2)}$ might arise if intervals do not intersect. However, Assumption 5 rules this problem out. Thus, for any observable payoffs $x_1, x_2 \in X_S$, the ratio $\frac{u'(x_1)}{u'(x_2)}$ is identified. This implies that the utility function $u(\cdot)$ is identified up to affine transformation on X_S.

From the results of Lemma 7, all boundaries are classified by stages. If two regions are separated by the conditional stage boundary related to event A, then $P_{\alpha,x}(A|\pi)$ for all $A \in \pi$ and $P_{\alpha,x}(s|B)$ for all $B \in \pi \backslash A$ stay the same, while only $P_{\alpha,x}(s|A)$ change. If regions are separated by the ex-ante stage boundary, then all probabilities change. By looking at which probabilities change from one region to another, we can identify the partition.

By denoting different neighboring regions through x_i and x_j, one can obtain:

$$\frac{P_{\alpha;x_i}(s|A)}{P_{\alpha;x_j}(s|A)} = \frac{P_{x_i}(s|\pi)}{P_{x_j}(s|\pi)}.$$

If k is a number of different conditional stage regions for event A, and n is a number of states in event A, then due to full-dimensionality, $k \geq n$. Because all ratios $\frac{P_{\alpha;x_i}(s|A)}{P_{\alpha;x_j}(s|A)}$ for conditional stage regions are known and $\sum_{s \in A} P_{\alpha;x_i}(s|A) = 1$, we have kn unknowns and $n(k-1)+k$ equations. Because $n(k-1)+k \geq kn$, all $P_{\alpha;x_i}(s|A)$ are identified for all i.

Identification of $P_{\alpha;x}(A|\pi)$ is trivial now:

$$P_{\alpha;x}(A|\pi) = \frac{P_x(s|\pi)}{P_{\alpha;x}(s|A)}.$$

$P_{\min x}(\cdot)$ and $P_{\max x}(\cdot)$ are solutions to the linear programming problem at each stage. However, in symmetric regions, $P_{\min x}(\cdot)$ and $P_{\max x}(\cdot)$ will exchange places due to the direction of utility growth, which will be the opposite. Thus, if we denote each pair of these probabilities as (x_s, y_s)

for conditional priors, we will obtain two equations:

$$\alpha x_s + (1-\alpha)y_s = P_{\alpha;x_1}(s|A)$$
$$\alpha y_s + (1-\alpha)x_s = P_{\alpha;x_2}(s|A),$$

where x_1 and x_2 are acts from symmetric regions. We can write down such systems for all states and all symmetric regions for conditional probabilities. In addition, we can do the same for the ex-ante probabilities:

$$\alpha x_A + (1-\alpha)y_A = P_{\alpha;x_1}(A|\pi)$$
$$\alpha y_A + (1-\alpha)x_A = P_{\alpha;x_2}(A|\pi).$$

In the final system of equations, there is always one unknown variable more than the number of equations; thus, the system does not have a unique solution.

First, note that $\alpha = 0.5$ if and only if $P_{\alpha;x_1}(s|A) = P_{\alpha;x_2}(s|A)$ and $P_{\alpha;x_1}(A|\pi) = P_{\alpha;x_2}(A|\pi)$ for all symmetric regions, states, and events. Thus, we can always identify whether $\alpha = 0.5$ or $\alpha \neq 0.5$. In addition, it is possible to identify whether the agent is ambiguity-averse/-loving. If we fix bundle payoffs at all states except $s_i, s_j \in A \in \pi$ and move along one IC from high x_i and low x_j to low x_i and high x_j inside one ex-ante region, then, if $\alpha > 0.5$, $\frac{P_{\alpha;x}(s_j|A)}{P_{\alpha;x}(s_i|A)}$ will decrease, while for $\alpha < 0.5$ the probability ratio will increase.

Consider the case of ambiguity aversion. Note that $\alpha = 1$ is always possible. In addition, there always exists the lowest possible value, $\underline{\alpha} > 0.5$, for α that satisfies the system of equations. Then, due to the bilinearity of the system, $\mathcal{A} = [\underline{\alpha}, 1)$. In addition, the greater the value of α, the smaller are the sets of priors $\mathcal{P}(A)$ and $\mathcal{P}(\pi)$. Thus, the sets of priors $\widetilde{\mathcal{P}}(A)$ and $\widetilde{\mathcal{P}}(\pi)$ will correspond to $\underline{\alpha}$ in the above system of equations.

By analogy, if $\alpha < 0.5$, then there exists an upper bound $\bar{\alpha}$ such that $\mathcal{A} = (0, \bar{\alpha}]$, and $\widetilde{\mathcal{P}}(A)$ and $\widetilde{\mathcal{P}}(\pi)$ are sets of priors that correspond to $\bar{\alpha}$.

□

Proof of Proposition 5. The proof is similar to the proof of Proposition 1 in Polisson, Quah, and Renou (2015).

For sufficiency, we first denote the following sets: for each $A \in \pi$, Q_A is a convex hull of $\{q_t^A(x)\}_{t=1,2}$; W is a convex hull of $\{w_t(x)\}_{t=1,2}$ for all $x \in \mathcal{L}$. Now we show that $q_1^A(x)\bar{u}(x) = \min_{q \in Q_A} q\bar{u}(x)$. Suppose that it's not true – i.e., there exists $\tilde{x} \in \mathcal{L}$ and $q \in Q_A$ such that $q\bar{u}(\tilde{x}) < q_1^A(\tilde{x})\bar{u}(\tilde{x})$. Q_A is a convex hull of $\{q_t^A(x)\}_{t=1,2}$; thus, q is a convex combination of $\{q_t^A(x)\}_{t=1,2}$. This means that there exists $x' \in \mathcal{L}$ such that $q_t^A(x')\bar{u}(\tilde{x}) < q_1^A(\tilde{x})\bar{u}(\tilde{x})$ – i.e., a contradiction. By analogy, one can show that $q_2^A(x)\bar{u}(x) = \max_{q \in Q_A} q\bar{u}(x)$. Thus, $\mu_\alpha(x)\bar{u}(x) = a \min_{q \in Q_A} q\bar{u}(x) + (1-a) \max_{q \in Q_A} q\bar{u}(x)$.

Note, also, that

$$\sum_{A \in \pi} w_{1;A}(x) \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s) = \min_{w \in W} \sum_{A \in \pi} w \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s)$$

$$\sum_{A \in \pi} w_{2;A}(x) \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s) = \max_{w \in W} \sum_{A \in \pi} w \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s).$$

Thus,

$$\mu(x)\bar{u}(x) = a \min_{w \in W} \sum_{A \in \pi} w \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s) + (1-a) \max_{w \in W} \sum_{A \in \pi} w \sum_{s \in A} \mu_{\alpha;s}(x)\bar{u}(x_s).$$

Now, we define $\phi : \mathbb{R}_+^\Omega \to \mathbb{R}$ as

$$\phi(\bar{u}) = a \min_{w \in W} w \circ \left(a \min_{q \in Q_A} q\bar{u} + (1-a) \max_{q \in Q_A} q\bar{u} \right) + (1-a) \max_{w \in W} w \circ \left(a \min_{q \in Q_A} q\bar{u} + (1-a) \max_{q \in Q_A} q\bar{u} \right).$$

We can apply Theorem 1 from Polisson, Quah and Renou (2015) to guarantee the existence of u that rationalizes the dataset.

Note that necessity follows straightforward from the model.

□