On Machina's Paradoxes and Limited Attention

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Limited attention and similarity of some of the states of the world together may nudge an agent to perceive the "grand world" as a collection of "small worlds." We use this idea as an explanation for some of the ambiguity paradoxes posed by Machina (2009, 2014) as a challenge to the prominent ambiguity theories.

Keywords: limited attention, ambiguity paradoxes, state aggregation

1 Introduction

Economists have proposed a number of models to address the problem with subjective expected utility maximization when agents face ambiguity, that is situations with unknown probabilities of the states (Ellsberg (1961)). The major models include the following: Choquet expected utility (Schmeidler (1989)) assumes nonadditive probability priors. Maxmin expected utility (Gilboa & Schmeidler (1989)) assumes the existence of multiple priors and the agent who maximizes the expected utility with the worst of them. α-maxmin (Ghirardato et al. (2004)) is an extension of the maxmin model to a linear combination of the worst and best priors. The smooth model of ambiguity aversion (Klibanoff et al. (2005)) considers the agent with the set of beliefs over possible sets of priors over ambiguous states and proposes recursive non-linear

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evaluation of the expected utility. Finally, variational preferences (Maccheroni et al. (2006)) extend the maxmin model to include the cost function that represents the ambiguity attitude. However, Machina (2009, 2014) proposed a number of theoretical challenges to the Choquet theory and to some of the other ambiguity models. Furthermore, some of the offered ambiguity paradoxes have been confirmed in experimental settings (L'Haridon & Placido (2010), Aerts, Geriente, Moreira & Sozzo (2018), Chen & Schonger (2015)).

The existing literature suggests different explanations for the phenomena. According to Machina, in the cases of three or more states, all major ambiguity models allow substantial separability between states in the value functional. Hence, a model attempting to explain the phenomena must restrict the separability in some way. Dillenberger & Segal (2015) employ the recursive ambiguity model of Segal (1987) that, similar to the smooth ambiguity, assumes a distribution of priors over possible probability distributions, but using a non-expected utility functional in the first stage – the main source of non-separability. In particular, the authors construct their examples with the model of disappointment aversion (Gul (1991)). Aerts, Geriente, Moreira & Sozzo (2018) consider the quantum-theoretic framework of Aerts & Sozzo (2016) and Aerts, Haven & Sozzo (2018). In this model, the agent has several cognitive types that induce different quantum probability distributions over the states, and the agent might transition between her types. The lack of separability is the result of the usage of the quantum probability distributions.

By contrast, we look at the paradoxes from a different angle: To deal with a decision problem, the agent combines similar states together to form a relevant "small world." The structure of the "small worlds" is a partition of the state space, which is basically an attention filter. In our demonstration, we use the most basic model of the "small worlds" – the state aggregation subjective expected utility (SASEU) of Burkovskaya (2017). Similarly to Segal (1987), SASEU also exploits the recursive evaluation; however, the concept is different: The agent first evaluates expected values of the lotteries in each "small world," then applies some non-linear positive transformation to those values, and finally evaluates the expected value over the events in the partition of the state space. "Small worlds" together with a non-linear transformation allow restriction of the separability between payoffs in different states and thus explain some of the Machina's ambiguity paradoxes. Note that we use expected utility as the stage model; however, other models might allow for even more flexibility in providing the explanation.

The structure of the paper is as follows. Section 2 summarizes the SASEU model. In section 3, we demonstrate the results that SASEU delivers in Machina examples. Section 4 concludes.

2 State Aggregation Subjective Expected Utility

In this section, we outline the SASEU model of Burkovskaya (2017) that we use in the analysis that follows.

Suppose $\Omega = \{s_1, s_2, \ldots, s_n\}$ is a finite set of states of the world and π is a partition of Ω. In this model, different partitions represent various ways of aggregating similar states into "small worlds." Let $x = (x_1, s_1; x_2, s_2; \ldots; x_n, s_n)$ represent a lottery paying x_i in state s_i. The agent has a unique subjective (or objective) prior $p = (p_1, \ldots, p_n)$ about the probabilities of different states. Given state aggregation π, for each event $A \in \pi$, the agent evaluates corresponding conditional probabilities $P(s|A) = \frac{p_s}{\sum_{s\in A} p_s}$ and uses the following value functional to evaluate lottery x:

$$V_\pi(x) = \sum_{A\in\pi} P(A)\phi\left(\sum_{s\in A} P(s|A)u(x_s)\right).$$

Note $V_\pi(x) = \sum_{A\in\pi} P(A)V_A(x)$ is a regular SEU functional, whereas $V_A(x) = \phi\left(\sum_{s\in A} P(s|A)u(x_s)\right)$ is a positive transformation of a conditional-stage SEU functional. If either ϕ is linear or $\pi \in \{\Omega, \{\Omega\}\}$, the model simplifies to the regular

SEU.

Example: Consider an agent in the Ellsberg world, which has three states: risky red ball, ambiguous yellow ball, and ambiguous black ball, i.e., $\Omega = \{R, Y, B\}$. The probability of the risky red ball (R) is $\frac{1}{3}$, whereas the probability of the yellow ball (Y) and the probability of the black ball (B) are unknown. The agent is offered a lottery x. The agent sees a similarity between yellow- and black-ball states, and because of the limited attention, aggregates these states into a "small world" called "ambiguous." Hence, the whole outcome space consists of events "risky" R and "ambiguous" YB. We denote the state aggregation in this case as $\pi = \{R, YB\}$. No particular difference exists between yellow and black balls, and we have no reason to believe one of the states is more likely. Thus, because of this symmetry, the agent forms a subjective belief that $P(Y|YB) = P(B|YB) = 0.5$. Then, the value functional for lottery x would be

$$V_\pi(x) = \frac{1}{3}\phi(u(x_R)) + \frac{2}{3}\phi\left(0.5u(x_Y) + 0.5u(x_B)\right).$$

Lottery	R	Y	B
f_1	\$100	0	0
f_2	0	0	\$100
f_3	\$100	\$100	0
f_4	0	\$100	\$100

Table 1: Ellsberg lotteries

In the Ellsberg experiment, the subjects are offered a choice between lotteries f_1 and f_2 and between f_3 and f_4 in Table 1. Let us demonstrate how an SASEU-maximizer evaluates the above lotteries in this case:

$$V_\pi(f_1) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(u(0)\right)$$
$$V_\pi(f_2) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.5u(100) + 0.5u(0)\right).$$

Notice $V_\pi(f_1) > V_\pi(f_2)$ if and only if

$$0.5\phi(u(100)) + 0.5\phi(u(0)) > \phi\left(0.5u(100) + 0.5u(0)\right). \tag{1}$$

The other two lotteries are evaluated as follows:

$$V_\pi(f_3) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.5u(100) + 0.5u(0)\right)$$
$$V_\pi(f_4) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(u(100)\right).$$

Moreover, $V_\pi(f_4) > V_\pi(f_3)$ if and only if eq. (1) holds.

3 Ambiguity Paradoxes

In this section, we go over well-known Machina's thought experiments that demonstrate the problems that the Choquet and in some cases other prominent ambiguity theories have in explaining behavior.

3.1 Reflection Example

Machina (2009) introduced the following thought experiment. An urn contains 100 balls; half are marked with either 1 or 2, and the other half are marked with either 3 or 4. One ball is drawn at random. The subject is offered the choice between lotteries f_5 and f_6 and between f_7 and f_8, payoffs for which are shown in Table 2.

The Choquet model implies $f_5 \succ f_6$ if and only if $f_7 \succ f_8$, because f_7 and f_8 are obtained from f_5 and f_6 by switching 0 and \$4,000. Machina argues that no difference exists between lotteries f_5 and f_8 and between f_6 and f_7, implying $f_5 \succ f_6$ if and only if $f_8 \succ f_7$. Thus, only indifference between the four lotteries would be possible. The experimental evidence is somewhat mixed. L'Haridon & Placido (2010) found that over 70% of the subjects choose f_6 and f_7. On the other hand, Aerts, Geriente,

	50 balls		50 balls	
Lottery	E_1	E_2	E_3	E_4
f_5	\$4,000	\$8,000	\$4,000	0
f_6	\$4,000	\$4,000	\$8,000	0
f_7	0	\$8,000	\$4,000	\$4,000
f_8	0	\$4,000	\$8,000	\$4,000

Table 2: Reflection Example

Moreira & Sozzo (2018) discovered that 29.5% of the subjects chose f_5 with f_8, 32% chose f_6 with f_7, and the rest of the subjects showed preference reversal, which might be indicative of the indifference. In addition, theory-wise, Baillon et al. (2011) demonstrated that MEU, variational preferences, and the smooth ambiguity model do not allow $f_6 \succ f_5$ and $f_7 \succ f_8$. The authors also showed that the preferences might be represented by α-MEU; however, this would require unreasonable priors.

An SASEU maximizer will aggregate states E_1 and E_2 into event E_{12} and states E_3 and E_4 into event E_{34}. The probabilities of E_1 and E_2 given event E_{12} have no reason to be different, so $P(E_1|E_{12}) = P(E_2|E_{12}) = 0.5$. By analogy, $P(E_3|E_{34}) = P(E_4|E_{34}) = 0.5$. The probabilities of E_{12} and E_{34} are objective and equal 0.5. Thus, the values of the lotteries are

$$V_\pi(f_5) = V_\pi(f_8) = 0.5\phi\left(0.5u(4,000) + 0.5u(8,000)\right) + 0.5\phi\left(0.5u(4,000) + 0.5u(0)\right)$$
$$V_\pi(f_6) = V_\pi(f_7) = 0.5\phi\left(u(4,000)\right) + 0.5\phi\left(0.5u(8,000) + 0.5u(0)\right).$$

Notice a concave $\phi(\cdot)$ implies $f_6 \succ f_5$ and $f_7 \succ f_8$, whereas convex $\phi(\cdot)$ implies $f_5 \succ f_6$ and $f_8 \succ f_7$.

3.2 Slightly Bent Coin Problem

Machina (2014) offers the following thought experiment: An agent needs to choose between two bets. The payout depends on a flip of a slightly bent coin (in which direction the coin is bent is unknown) and the color of the ball drawn from the urn. The urn contains two balls, each of which can be black or white. Hence, the state space consists of four states dependent on whether the ball is black or white and whether the coin lands heads or tails, i.e., $\Omega = \{BH, BT, WH, WT\}$. The bets are in Table 3.

I	Black	White	II	Black	White
Heads	8,000	0	Heads	0	0
Tails	-8,000	0	Tails	-8,000	8,000

Table 3: Slightly Bent Coin Bets

Machina argues the Choquet expected utility predicts indifference in this case, whereas if the coin is only slightly bent, the ambiguity-averse consumer should choose bet I over bet II.

Consider an SASEU maximizer who aggregates states with different balls and the same coin together. Thus, states BH and WH are aggregated into event H, and states BT and WT are aggregated into event T, forming a subjective partition $\pi = \{H, T\}$. First, we have no reason to believe that probabilities of white and black balls differ, so $P(W|H) = P(B|H) = P(W|T) = P(B|T) = 0.5$. In the same manner, we have no reason to believe the coin is bent in a specific direction and the probabilities of heads and tails differ, implying $P(H) = P(T) = 0.5$. The value of each bet can be calculated as

$$V_\pi(I) = 0.5\phi\left(0.5u(0) + 0.5u(-8,000)\right) + 0.5\phi\left(0.5u(0) + 0.5u(8,000)\right)$$
$$V_\pi(II) = 0.5\phi\left(0.5u(8,000) + 0.5u(-8,000)\right) + 0.5\phi(u(0)).$$

Note concave $\phi(\cdot)$ predicts $I \succ II$, whereas convex $\phi(\cdot)$ predicts $II \succ I$.

3.3 Ambiguity at Low- vs. High-Outcomes Problem

Machina (2014) also proposed this paradox. The subject is asked to choose between two urns. Both urns contain three balls, one of which is known to be red. Each of the other balls can be either black or white. The value of c is defined as the certainty equivalent of a 50:50 bet for \$0 and \$100. The payoffs of the urns are shown in Table 4.

Urn	R	B	W
I	\$100	\$0	\$$c$
II	\$0	\$$c$	\$100

Table 4: Ambiguity at Low vs. High Outcomes

The Choquet theory predicts indifference between the urns. However, Machina argues the subjects might prefer urn II. Chen & Schonger (2015) found that approximately 60% of the subjects chose urn I.

An SASEU agent might naturally want to aggregate states B and W into "ambiguous" event BW for both urns. Then, probabilities of the events R and BW are objective: $P(R) = \frac{1}{3}$ and $P(BW) = \frac{2}{3}$. In addition, given event BW, we have no reason to believe the probabilities of B and W are different for any of the urns; thus, $P(B|BW) = P(W|BW) = 0.5$. Hence, the value of each urn is as follows:

$$V_\pi(I) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.5u(c) + 0.5u(0)\right) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.25u(100) + 0.75u(0)\right)$$
$$V_\pi(II) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.5u(c) + 0.5u(100)\right) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.75u(100) + 0.25u(0)\right).$$

Notice that depending on $\phi(\cdot)$ and $u(\cdot)$, any behavior might be obtained. For example, if $\phi(x) = \sqrt{x}$, $u(0) = 0$ and $u(100) = 1$, then $I \succ II$; however, if $\phi(x) = x^{1.5}$, then $II \succ I$. We do not find any particular relationship between the choices and concavity/convexity of $\phi(\cdot)$ in this example.

4 Conclusion

State aggregation is an attention filter that nudges an agent to form "small worlds" from similar states, and as a result naturally restricts the separability between those states in the value functional. This paper has shown how limited attention might be used to explain a number of ambiguity paradoxes of Machina (2009, 2014), who argued the failure of Choquet and other ambiguity models derives primarily from too much separability between the corresponding states.

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